A Report On:

A Unified Formulation of the Segregated Class of Algorithms For Multi-Fluid Flow at All-Speeds

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by

F. Moukalled and M. Darwish
American University of Beirut,
Faculty of Engineering & Architecture,
Mechanical Engineering Department,
P.O.Box 11-0236
Beirut - Lebanon

Address all correspondence to Dr. F. Moukalled,
email: memouk@aub.edu.lb

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Professor Fadl Moukalled

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American University of Beirut
ME Dept. PO Box: 11-0236
Beirut
Lebanon

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This report results from a contract tasking American University of Beirut as follows: It has been determined that the segregated class of algorithms and the many techniques developed for single-fluid flow, has not been fully extended to multi-fluid flows. Thus the main objectives of this work are: 1) To extend the applicability of the available segregated class of algorithms for single-fluid flow at all speeds to multi-fluid flow at all speeds; 2) To derive these algorithms using a unified, compact, and easy to understand notation that can be expanded systematically to yield the coefficients of the pressure correction equation, thus facilitating the implementation of these algorithms to a wider audience within the Computational Fluid Dynamics (CFD) community; 3) To explain the philosophies behind these algorithms and to pinpoint their similarities and differences.

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Abstract

In this paper, the class of segregated single-fluid all speed flow algorithms is extended to multi-fluid flow simulations using a unified, compact, and easy to understand notation. Depending on the constraint equation used to derive the pressure correction equation, the extended multi-fluid flow algorithms are shown to fall under two categories denoted in this work by the geometric conservation based algorithms and the mass conservation based algorithms, respectively. The differences and similarities between the two categories are explained. Several techniques developed to promote and accelerate the convergence of these algorithms are also presented.

Nomenclature

$A_P^{(k)}$,..	coefficients in the discretized equation for $\phi^{(k)}$.
$B_P^{(k)}$	source term in the discretized equation for $\phi^{(k)}$.
$\mathbf{B}^{(k)}$	body force per unit volume of fluid/phase k.
$C_\rho^{(k)}$	coefficient equals to $1/R^{(k)}T^{(k)}$.
$\hat{\mathbf{d}}_f$	covariant unit vector (i.e. in the direction of $\mathbf{d}_f$).
$D_P^{(k)}[\phi^{(k)}]$	the D operator.
$\tilde{D}_P^{(k)}[\phi^{(k)}]$	the modified D operator.
$\mathbf{D}_P^{(k)}[\phi^{(k)}]$	the vector form of the D operator.
$\tilde{\mathbf{D}}_P^{(k)}[\phi^{(k)}]$	the vector form of the modified D operator
$H_P[\phi^{(k)}]$	the H operator.
$HI_P[\phi^{(k)}]$	the *HI* operator working on $\phi^{(k)}$ ($\phi^{(k)}$=$u^{(k)}$,$v^{(k)}$, or $w^{(k)}$)
$HP_P[\phi^{(k)}]$	the *HP* operator working on $\phi^{(k)}$ ($\phi^{(k)}$=$u^{(k)}$,$v^{(k)}$, or $w^{(k)}$)
$\tilde{HP}_P[\phi^{(k)}]$	the modified *HP* operator working on $\phi^{(k)}$ ($\phi^{(k)}$=$u^{(k)}$,$v^{(k)}$, or $w^{(k)}$)
$\mathbf{HP}_P[\mathbf{u}^{(k)}]$	the vector form of the *HP* operator.
$\tilde{\mathbf{HP}}_P[\mathbf{u}^{(k)}]$	the vector form of the modified *HP* operator.
$\mathbf{HI}_P[\mathbf{u}^{(k)}]$	the vector form of the *HI* operator.
$\mathbf{i}$	unit vector in the x-direction.
$\mathbf{I}^{(k)}$	inter-phase momentum transfer.
$\mathbf{j}$	unit vector in the y-direction.
$\mathbf{J}_f^{(k)}$	total flux of $\phi^{(k)}$ across cell face ‘f’.
$\mathbf{J}_f^{(k)D}$	diffusion flux of $\phi^{(k)}$ across cell face ‘f’.

$\mathbf{J}_f^{(k)C}$	convection flux of $\phi^{(k)}$ across cell face 'f'.
$\dot{M}^{(k)}$	source of mass per unit volume.
$\tilde{\mathrm{n}}_f$	contravariant unit vector (i.e. in the direction of $\mathbf{S}_\mathrm{f}$).
P	pressure.
$\dot{q}^{(k)}$	heat generated per unit volume of fluid/phase k.
$Q^{(k)}$	general source term of fluid/phase k.
$r^{(k)}$	volume fraction for fluid/phase k.
$R^{(k)}$	gas constant for fluid/phase k.
$\mathbf{S}_\mathrm{f}$	surface vector.
t	time.
$T^{(k)}$	temperature of fluid/phase k.
$U_f^{(k)}$	interface flux velocity $(\mathbf{v}_f^{(k)}.\mathbf{S}_f$ of fluid/phase k.
$\mathbf{u}^{(k)}$	velocity vector of fluid/phase k.
$\mathrm{u}^{(k)},\mathrm{v}^{(k)},..$	velocity components of fluid/phase k.
x, y,z	Cartesian coordinates.
$\lVert \mathrm{a}, \mathrm{b} \rVert$	the maximum of a and b.

Greek Symbols

$\rho^{(k)}$	density of fluid/phase k.
$\Gamma^{(k)}$	diffusion coefficient of fluid/phase k.
$\Phi^{(k)}$	dissipation term in energy equation of fluid/phase k.
$\phi^{(k)}$	general scalar quantity associated with fluid/phase k.
$\phi_f^{(k)}$	scalar value at cell face 'f'.

κ_f	space vector equal to $\left(\hat{\mathbf{n}}_f - \gamma\hat{\mathbf{d}}_f\right)S_f$
$\Delta_F\left[\phi^{(k)}\right]$	the Δ operator.
$\mu^{(k)}$	viscosity of fluid/phase k.
Ω	cell volume.
$\beta^{(k)}$	thermal expansion coefficient for phase/fluid k.
δt	time step.
γ	scaling factor.

Subscripts

e, w, .	refers to the east, west, … face of a control volume.
E,W,..	refers to the East, West, … neighbors of the main grid point.
f	refers to control volume face f.
F	refers to main grid point F.
P	refers to the P grid point.

Superscripts

C	refers to convection contribution.
D	refers to diffusion contribution.
(k)	refers to fluid/phase k.
$(k)^*,(k)^{**},..$	refers to first, second, … updated value at the current iteration.
$(k)\circ$	refers to values of fluid/phase k from the previous iteration.
$(k)'$	refers to correction field of phase/fluid k.
old	refers to values from the previous time step.

sx refers to SIMPLEX.

x,y,z refers to components in x, y, and z directions.

Introduction

Over the past two decades important advances have taken place in CFD centered around increasing numerical accuracy through the development of high-resolution schemes [1,2,3,4,5,6,7,8,9,10,11,12,13,14], and improving efficiency through devising better solution algorithms [15,16,17,18,19,20,21], better solvers [22,23], and increasing use of multigrid techniques [24,25,26,27]. While high-resolution schemes, solvers, and multigrid techniques can be applied indiscriminately to the simulation of single-fluid or multi-fluid flows, nearly all developments in solution algorithms have been directed towards the simulation of incompressible, compressible, and more recently all-speed single-fluid flows [25,27,28,29,30,31,32]. In this paper, a solution algorithm denotes the procedure used to solve the coupling between the velocity, pressure, mass fraction, and density (for compressible flow) fields.

For the solution of single-fluid flow, a number of segregated solution algorithms have been developed such as the well-known SIMPLE [15,16], the SIMPLER [16], the SIMPLEST [33], the SIMPLEC [18], the SIMPLEM [34], the PISO [17], and the SIMPLEX [20] algorithms, to site a few. Additionally, several techniques have been advertised to improve the performance, facilitate the implementation, and extend the capability of these algorithms. The use of the Momentum Weighted Interpolation Method (MWIM) [35,36,37,38,39] and the Pressure Weighted Interpolation Method (PWIM) [37,40,41,42,43,44] that have enabled the implementation of these algorithms with a collocated variable arrangement is an example of such techniques. Another example is the extension of the segregated pressure-based algorithms to simulating compressible and all speed flows that span the entire subsonic to hypersonic spectrum [28,29,21,31,45,46,47,48].

On the other hand the development of multi-fluid solution algorithms has lagged behind that

of single-fluid algorithms, partly because of the much higher computational cost involved, and partly due to the numerical difficulties that had to be first addressed in the simulation of single-fluid flow. While the main difficulty in simulating single-fluid flow stems from the coupling between the momentum and continuity equations, in multi-fluid flow situations, this problem is further complicated by the fact that there are as many sets of continuity and momentum equations as there are fluids, that they are all coupled together in various ways (interchange momentum by interphase mass and momentum transfer,...), and that the fluids share space (the volume fractions sum to unity, but are not known in advance).

Despite these complexities, successful segregated pressure-based solution algorithms have been devised. These algorithms can be divided into two groups based on the constraint equation used in deriving the pressure or pressure correction equation. Among these algorithms is the IPSA algorithm (Inter-Phase Slip Algorithm) and its variants devised by the Spalding Group at Imperial College [49,50,51,52] and the Implicit Multi-Field [53] based algorithms (IMF) developed by the Los Alamos Scientific Laboratory (LASL) group [54,55,56,57,58,59,60,61,62]. However, in contrast with the widespread information available on single-fluid solution algorithms, much less information is available about multi-fluid solution algorithms, a fact that has confined their implementation to a small community, slowed their development, and isolated them from the newer developments in single-fluid flow algorithms (PWIM, all speed flows,...).

From the above, it is obvious that the segregated class of algorithms and the many techniques developed for single-fluid flow, have neither been fully extended nor applied to the simulation of multi-fluid flow. The main objective of this work is to extend the applicability of single-fluid algorithms to multi-fluid flow simulations, and to derive these algorithms using a unified, compact, and easy to understand notation that can be expanded systematically to yield the coefficients of the pressure correction equation, thus facilitating

the implementation of these algorithms to a wider audience within the CFD community. In the process, the philosophies behind these algorithms in addition to their similarities and differences are explained. To this effect, it is shown that all multi-fluid algorithms can be implemented using a two-step procedure, whereby in the first step a multi-fluid pressure or pressure correction equation is derived based either on Volume Conservation (overall volume fraction equation) [49,63,64,65] or Mass Conservation (overall continuity equation) [66,50,63,64,67,68,69,70]. Then, in the second step, the different segregated single fluid algorithms are applied, to the constraint equation, yielding a correction equation that drives the global iterations towards convergence in a manner similar to the pressure correction equation in single-fluid flows.

In what follows, the governing equations for multi-fluid flows are presented and their discretization outlined so as to lay the ground for the derivation of the pressure or pressure-correction equation. Then, using the unified notation, the different multi-fluid solution algorithms are described and the framework for implementing them explained. However, it should be stressed that the intention of the paper is not to compare the relative performance of the different multi-fluid algorithms, this would be a work in progress, rather, the aim is to unify their formulation.

The Governing Equations

In multi-fluid flows the various components coexist with different concentrations at different locations in the flow domain and move with unequal velocities. Thus, the equations governing multi-fluid flows are the conservation laws of mass, momentum, and energy for each individual fluid. Moreover, the problem specification is completed by the introduction of auxiliary relations.

Conservation of mass

The volume fraction $r^{(k)}$, which is the proportion of volumetric space occupied by the k^{th} fluid ($\Omega^{(k)}/\Omega$) along with the k^{th} fluid density, $\rho^{(k)}$, and velocity, $\mathbf{u}^{(k)}$, in order to satisfy the mass-conservation principle, have to obey the differential equation:

$$\frac{\partial\left(r^{(k)}\rho^{(k)}\right)}{\partial t}+\nabla.\left(r^{(k)}\rho^{(k)}\mathbf{u}^{(k)}\right)=r^{(k)}\dot{M}^{(k)} \tag{1}$$

Mass sources are often non-zero, as when one fluid is transformed to another fluid. However, summation over all fluids leads to the "overall mass-conservation" equation:

$$\sum_k\left(\frac{\partial\left(r^{(k)}\rho^{(k)}\right)}{\partial t}+\nabla.\left(r^{(k)}\rho^{(k)}\mathbf{u}^{(k)}\right)\right)=0 \tag{2}$$

The zero on the right-hand side signifies that the sum of mass sources (generation and loss) is zero. Thus

$$\sum_k\left(r^{(k)}\dot{M}^{(k)}\right)=0 \tag{3}$$

The mass conservation equations can also be thought of as volume fraction equations (i.e. the equations used to calculate the volume fractions occupied by the various phases in the control volume). In this case, the equation can be rewritten as:

$$\frac{\partial\left(\rho^{(k)}r^{(k)}\right)}{\partial t}+\nabla.\left(\rho^{(k)}\mathbf{u}^{(k)}r^{(k)}\right)=\dot{M}^{(k)}r^{(k)} \tag{4}$$

Conservation of momentum

Denoting the velocity of the k^{th} phase by $\mathbf{u}^{(k)}$, then the momentum equation for the k^{th} phase can be written as:

$$\frac{\partial\left(r^{(k)}\rho^{(k)}\mathbf{u}^{(k)}\right)}{\partial t}+\nabla.\left(r^{(k)}\rho^{(k)}\mathbf{u}^{(k)}\mathbf{u}^{(k)}\right)=\nabla.\left(r^{(k)}\mu^{(k)}\nabla\mathbf{u}^{(k)}\right)+r^{(k)}\left(-\nabla P+\mathbf{B}^{(k)}\right)+\mathbf{I}^{(k)} \tag{5}$$

Here P stands for the pressure, which is regarded as being shared amongst the fluids, $\mathbf{B}^{(k)}$ is the body force per unit volume of phase (k), $\mathbf{I}^{(k)}$ is the momentum transfer to phase (k) resulting from interaction with other phases. The latter term vanishes when the "overall momentum equation" is formed. Thus:

$$\sum_k \left\{ \frac{\partial\left(r^{(k)}\rho^{(k)}\mathbf{u}^{(k)}\right)}{\partial t} + \nabla.\left(r^{(k)}\rho^{(k)}\mathbf{u}^{(k)}\mathbf{u}^{(k)}\right) \right\} = \sum_k \left\{ \nabla.\left(r^{(k)}\mu^{(k)}\nabla\mathbf{u}^{(k)}\right) + r^{(k)}\left(-\nabla P + \mathbf{B}^{(k)}\right) + \mathbf{I}^{(k)} \right. \quad (6)$$

The inter-phase momentum transfer accounts for the interactions among the fluids and can be written in the following form

$$\mathbf{I}^{(k)} = \sum_{m=(phases \neq k)} g^{(km)}\left(\mathbf{u}^{(m)} - \mathbf{u}^{(k)}\right) \quad (7)$$

The inter-phase friction force causing the velocity slip is an internal force for the system as a whole.

Conservation of energy

Let $T^{(k)}$ be the temperature of the k^{th} phase, the energy equation for the k^{th} phase is given by:

$$\begin{aligned} \frac{\partial\left(r^{(k)}\rho^{(k)}T^{(k)}\right)}{\partial t} + \nabla.\left(r^{(k)}\rho^{(k)}\mathbf{u}^{(k)}T^{(k)}\right) &= \frac{1}{c_P^{(k)}}\nabla.\left(r^{(k)}k^{(k)}\nabla T^{(k)}\right) \\ &+ \frac{r^{(k)}}{c_P^{(k)}}\left\{\beta^{(k)}T^{(k)}\left[\frac{\partial P}{\partial t} + \nabla.\left(P\mathbf{u}^{(k)}\right) - P\nabla.\left(\mathbf{u}^{(k)}\right)\right] + \Phi^{(k)} + q^{(k)}\right\} \end{aligned} \quad (8)$$

where

$$\Phi^{(k)} = \mu^{(k)} \left\{ \begin{aligned} & 2\left[\left(\frac{\partial u^{(k)}}{\partial x}\right)^2 + \left(\frac{\partial v^{(k)}}{\partial y}\right)^2 + \left(\frac{\partial w^{(k)}}{\partial z}\right)^2\right] + \left(\frac{\partial u^{(k)}}{\partial y} + \frac{\partial v^{(k)}}{\partial x}\right)^2 \\ & + \left(\frac{\partial u^{(k)}}{\partial z} + \frac{\partial w^{(k)}}{\partial x}\right)^2 + \left(\frac{\partial v^{(k)}}{\partial z} + \frac{\partial w^{(k)}}{\partial y}\right)^2 - \frac{2}{3}\left(\nabla.\mathbf{u}^{(k)}\right)^2 \end{aligned} \right\} \quad (9)$$

and $\beta^{(k)}$ the thermal expansion coefficient of the k^{th} phase which is equal to $1/T^{(k)}$ for an ideal gas.

The General Multi-Fluid Scalar Equation

A review of the above differential equations reveals that they are similar in structure. If a typical representative variable associated with phase (k) is denoted by $\phi^{(k)}$, the general differential equation may be written as:

$$\frac{\partial\left(r^{(k)}\rho^{(k)}\phi^{(k)}\right)}{\partial t}+\nabla.\left(r^{(k)}\rho^{(k)}\mathbf{u}^{(k)}\phi^{(k)}\right)=\nabla.\left(r^{(k)}\Gamma^{(k)}\nabla\phi^{(k)}\right)+r^{(k)}Q^{(k)} \tag{10}$$

where the expression for $\Gamma^{(k)}$ and $Q^{(k)}$ can be deduced from the parent equations. The four terms in the above equation describe successively unsteadiness, convection (or advection), diffusion, and generation/dissipation effects. In fact all terms not explicitly accounted for in the first three terms are included in the catch-all source term $Q^{(k)}$.

Auxiliary Relations

The above set of differential equations has to be solved in conjunction with observance of constraints on the values of the variables represented by algebraic relations. These auxiliary relations express physical laws of various kinds:

a. Geometric Conservation Equation

The equation governing the volume fractions over a control volume can be written as

$$\sum_k r^{(k)}=1 \tag{11}$$

Physically, the geometric conservation equation is a statement indicating that the sum of volumes occupied by the different fluids within a cell is equal to the volume of the cell containing the fluids.

b. Equations of State

For every fluid element of a compressible multi-fluid flow, an auxiliary equation of state relating density to pressure and temperature is needed. For the k^{th} phase, such an equation is written as:

$$\rho^{(k)} = \rho^{(k)}\left(P, T^{(k)}\right) \tag{12}$$

In order to represent a complete mathematical problem, initial and boundary conditions expressing the particular situation to be investigated should supplement the above partial differential equations and auxiliary relations.

Discretization Procedure

In the previous sections the differential equations governing multi-fluid flow phenomena were presented as well as the associated auxiliary relations. The task now is to present the Finite Volume-based numerical solution algorithm for solving these equations [16].

Discretization of the General Conservation Equation

The general conservation equation (10) is integrated over a finite volume (Fig. 1) to yield the following expression:

$$\begin{aligned} &\iint_{\Omega} \frac{\partial\left(r^{(k)}\rho^{(k)}\phi^{(k)}\right)}{\partial t} d\Omega + \iint_{\Omega} \nabla .\left(r^{(k)}\rho^{(k)}\mathbf{u}^{(k)}\phi^{(k)}\right) d\Omega \\ &\qquad = \iint_{\Omega} \nabla .\left(r^{(k)}\Gamma^{(k)}\nabla\phi^{(k)}\right) d\Omega + \iint_{\Omega} r^{(k)} Q^{(k)} d\Omega \end{aligned} \tag{13}$$

where Ω is the volume of the control cell. Using the divergence theorem to transform the volume integral to a surface integral, one obtains:

$$\iint_{\Omega} \frac{\partial\left(r^{(k)}\rho^{(k)}\phi^{(k)}\right)}{\partial t} d\Omega + \oint_{S} \left(r^{(k)}\rho^{(k)}\mathbf{u}^{(k)}\phi^{(k)}\right)\cdot \mathbf{dS} = \oint_{S} \left(r^{(k)}\Gamma^{(k)}\nabla\phi^{(k)}\right)\cdot \mathbf{dS} + \iint_{\Omega} r^{(k)}Q^{(k)} d\Omega \tag{14}$$

Replacing the surface integral over the control volume by a summation of the flux terms over the sides of the control volume equation (14) is transformed to:

$$\begin{aligned}\frac{\partial\left(r^{(k)}\rho^{(k)}\phi^{(k)}\Omega\right)}{\partial t} &+ \sum_{nb=e,w,n,s,t,b} \left[\left(r^{(k)}\rho^{(k)}\mathbf{u}^{(k)}\phi^{(k)}\right)_{nb} \cdot \mathbf{S}_{nb}\right] \\ &= \sum_{nb=e,w,n,s,t,b} \left[\left(r^{(k)}\Gamma^{(k)}\nabla\phi^{(k)}\right)_{nb} \cdot \mathbf{S}_{nb}\right] + r^{(k)}Q^{(k)}\Omega\end{aligned} \tag{15}$$

It is worth noting that for an arbitrary subdivision of the volume Ω, into say n sub-volumes, the conservation equation over each control volume can be integrated and the global conservation equation recovered by adding up the n sub-volume conservation equations since the internal surface integral appears twice but with opposite signs and will cancel.

For an arbitrary quadrilateral (Fig. 2), equation (15) can be rewritten as:

$$\frac{\partial\left(r^{(k)}\rho^{(k)}\phi^{(k)}\Omega\right)}{\partial t} + \sum_{nb=e,w,n,s,t,b} \mathbf{J}_{nb}^{(k)} = r^{(k)}Q^{(k)}\Omega \tag{16}$$

where $\mathbf{J}_{nb}^{(k)}$ represents the total flux of ϕ across cell face nb and is given by

$$\mathbf{J}_{nb}^{(k)} = \left(r^{(k)}\rho^{(k)}\mathbf{u}^{(k)}\phi^{(k)} - r^{(k)}\Gamma^{(k)}\nabla\phi^{(k)}\right)_{nb} \cdot \mathbf{S}_{nb} \tag{17}$$

Each of the surface fluxes $\mathbf{J}_{nb}^{(k)}$ contains a convective contribution, $\mathbf{J}_{nb}^{(k)C}$, and a diffusive contribution, $\mathbf{J}_{nb}^{(k)D}$, hence:

$$\mathbf{J}_{nb}^{(k)} = \mathbf{J}_{nb}^{(k)D} + \mathbf{J}_{nb}^{(k)C} \tag{18}$$

The discretization of the diffusive and convective fluxes are presented along an east face of a control volume. Starting with the diffusive flux, its value along the *'east'* side is given by:

$$\begin{aligned}\mathbf{J}_{e}^{(k)D} &= \left(-r^{(k)}\Gamma^{(k)}\nabla\phi^{(k)} \cdot \mathbf{S}\right)_{e} = \left(-r^{(k)}\Gamma^{(k)}\nabla\phi^{(k)} \cdot \hat{\mathbf{n}}S\right)_{e} \\ &= -r_{e}^{(k)}\Gamma_{e}^{(k)}\left[\left(\nabla\phi^{(k)}\right)_{e}.\left(\gamma\hat{\mathbf{d}}\right)_{e} + \left(\overline{\nabla\phi^{(k)}}\right)_{e}.\left(\hat{\mathbf{n}}_{e} - \left(\gamma\hat{\mathbf{d}}\right)_{e}\right)\right]S_{e}\end{aligned} \tag{19}$$

where γ is a scaling factor defined as [71]:

$$\gamma_e = \frac{1}{\hat{\mathbf{n}}_e . \hat{\mathbf{d}}_e} = \frac{S_e d_e}{\mathbf{S}_e . \mathbf{d}_e} \tag{20}$$

$\hat{\mathbf{n}}_f$ and $\hat{\mathbf{d}}_f$ (Fig. 2) are the contravariant (surface vector) and covariant (curvilinear coordinate) unit vectors respectively, and the over bar denotes a value obtained by interpolation. Defining the space vector κ_e as:

$$\kappa_{\mathbf{e}} = \left(\hat{\mathbf{n}}_e - \left(\gamma \hat{\mathbf{d}}\right)_e\right) S_e = \kappa_e^x \mathbf{i} + \kappa_e^y \mathbf{j} + \kappa_e^z \mathbf{k} \tag{21}$$

the diffusion flux becomes:

$$\begin{aligned} \mathbf{J}_e^{(k)D} &= -r_e^{(k)} \Gamma_e^{(k)} \left[\frac{\left(\phi_E^{(k)} - \phi_P^{(k)}\right)}{d_e} \frac{S_e d_e}{\mathbf{S}_e . \mathbf{d}_e} S_e + \left(\overline{\nabla \phi^{(k)}}\right)_e . \kappa_e \right] \\ &= -r_e^{(k)} \Gamma_e^{(k)} \left[\left(\phi_E^{(k)} - \phi_P^{(k)}\right) \frac{\mathbf{S}_e . \mathbf{S}_e}{\mathbf{S}_e . \mathbf{d}_e} + \left(\overline{\nabla \phi^{(k)}}\right)_e \kappa_e \right] \end{aligned} \tag{22}$$

Similar expressions are obtained for other faces. The convective flux for the *'east'* side is given by:

$$J_e^{(k)C} = \left(r^{(k)} \rho^{(k)} \mathbf{u}^{(k)} \cdot \mathbf{S}\right)_e \phi_e^{(k)} = \left(r^{(k)} \rho^{(k)} U^{(k)}\right)_e \phi_e^{(k)} = C_e^{(k)} \phi_e^{(k)} \tag{23}$$

where $\mathbf{S}_e$ and C_e are the surface vector and convection flux coefficient at cell face 'e', respectively. As can be seen from Eq. (23), the accuracy of the control volume solution for the convective scalar flux depends on the proper estimation of the face value ϕ_e as a function of the neighboring node values. Using some assumed interpolation profile, ϕ_e can be explicitly formulated in terms of its node values by a functional relationship of the form:

$$\phi_e = f(\phi_{NB}, C_e) \tag{24}$$

where ϕ_{NB} denotes the neighboring node values (ϕ_E, ϕ_W, ϕ_N, ϕ_S, ϕ_T, ϕ_B, ϕ_P, ϕ_{EE}, ϕ_{WW}, ϕ_{NN}, ϕ_{SS}, ϕ_{TT}, ϕ_{BB}, etc...).

After substituting the face values by their functional relationship relating to the node values of ϕ, Eq. (16) is transformed after some algebraic manipulations into the following discretized equation:

$$A_P^{(k)}\phi_P^{(k)} = \sum_{NB} A_{NB}^{(k)}\phi_{NB}^{(k)} + B_P^{(k)} \tag{25}$$

where the coefficients $A_P^{(k)}$ and $\phi_{NB}^{(k)}$ depend on the selected scheme and $B_P^{(k)}$ is the source term of the discretized equation . In compact form, the above equation can be written as

$$\phi^{(k)} = H_P\left[\phi^{(k)}\right] = \frac{\sum_{NB} A_{NB}^{(k)}\phi_{NB}^{(k)} + B_P^{(k)}}{A_P^{(k)}} \tag{26}$$

Discretization of the momentum equation

The discretization procedure of the momentum equation yields a discretized equation of the form:

$$\mathbf{A}_P^{(k)}\mathbf{u}_P^{(k)} = \sum_{NB(P)} \mathbf{A}_{NB}^{(k)}\mathbf{u}_{NB}^{(k)} + \mathbf{B}_P^{(k)} - r_P^{(k)}\Omega_P\nabla_P(P) + \Omega_P \sum_{m=(phases\neq k)} g^{(km)}\left(\mathbf{u}_p^{(m)} - \mathbf{u}_p^{(k)}\right) \tag{27}$$

where:

$$\mathbf{u} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} \qquad \mathbf{A}_P^{(k)} = \begin{bmatrix} A_P^{(k),u} \\ A_P^{(k),v} \\ A_P^{(k),w} \end{bmatrix} \qquad \sum_{NB} \mathbf{A}_{NB}^{(k)}\mathbf{u}_{NB}^{(k)} = \begin{bmatrix} \sum_{NB} A_{NB}^{(k),u} u_{NB}^{(k)} \\ \sum_{NB} A_{NB}^{(k),v} v_{NB}^{(k)} \\ \sum_{NB} A_{NB}^{(k),w} w_{NB}^{(k)} \end{bmatrix} \tag{28}$$

$$\mathbf{B}_P^{(k)} = \begin{bmatrix} B_P^{(k),u} \\ B_P^{(k),v} \\ B_P^{(k),w} \end{bmatrix} \qquad \nabla_P(P) = \begin{bmatrix} \nabla_P(P).\mathbf{i} \\ \nabla_P(P).\mathbf{j} \\ \nabla_P(P).\mathbf{k} \end{bmatrix} = \begin{bmatrix} (\nabla P)_P^x \\ (\nabla P)_P^y \\ (\nabla P)_P^z \end{bmatrix} \tag{29}$$

In the above equation, the inter-phase term is written out explicitly to show the strong coupling among the momentum equations of the different fluids. This is different from the

spatial coupling that exists among the neighboring velocities of the same fluid. One way to improve the overall convergence and robustness of the algorithm is to re-write the discretized momentum equations for the various phases such that:

$$\mathbf{A}_P^{(k)}\mathbf{u}_P^{(k)} = \sum_{NB}\mathbf{A}_{NB}^{(k)}\mathbf{u}_{NB}^{(k)} + \mathbf{B}_P^{(k)} - r_P^{(k)}\Omega_P\nabla_P(P) + \Omega_P \sum_{m=(phases\neq k)} g^{(km)}\mathbf{u}_p^{(m)} \tag{30}$$

Where now

$$\mathbf{A}_P^{(k)} \leftarrow \mathbf{A}_P^{(k)} + \Omega_P \sum_{m=all\ fluids\neq k}\left(g_P^{(km)}\right) \tag{31}$$

For later reference, the value of $\mathbf{A}_P^{(k)}$ before the addition of the inter-phase terms will be denoted by $\tilde{\mathbf{A}}_P^{(k)}$. To this end, the discretized form of the momentum equation can be rewritten as:

$$\mathbf{u}_P^{(k)} = \mathbf{HP}_P\left[\mathbf{u}^{(k)}\right] - r^{(k)}\mathbf{D}_P^{(k)}\nabla_P(P) \tag{32}$$

where the body force and inter-phase terms are absorbed in the $\mathbf{B}_P^{(k)}$ source term within the $\mathbf{HP}_P\left[\mathbf{u}^{(k)}\right.$ term, or as

$$\mathbf{u}_P^{(k)} = \mathbf{HI}_P\left[\mathbf{u}^{(k)}\right] + \mathbf{D}_P^{(k)} \sum_{m=(phases\neq k)}\left(g^{(km)}\mathbf{u}_P^{(m)}\right) \tag{33}$$

where the body force and pressure gradient terms are absorbed in the $\mathbf{B}_P^{(k)}$ source term within the $\mathbf{HP}_P\left[\mathbf{u}^{(k)}\right.$ term. In equations (32) and (33), $\mathbf{D}_P^{(k)}, \mathbf{HP}_P\left[\mathbf{u}^{(k)}\right.$, and $\mathbf{HI}_P\left[\mathbf{u}^{(k)}\right.$ are given by,

$$\mathbf{D}_P^{(k)} = \begin{bmatrix} D_P^{(k)}[u] & 0 & 0 \\ 0 & D_P^{(k)}[v] & 0 \\ 0 & 0 & D_P^{(k)}[w] \end{bmatrix} = \begin{bmatrix} \dfrac{\Omega_P}{A_P^{(k),u}} & 0 & 0 \\ 0 & \dfrac{\Omega_P}{A_P^{(k),v}} & 0 \\ 0 & 0 & \dfrac{\Omega_P}{A_P^{(k),w}} \end{bmatrix} = \frac{\Omega_P}{\mathbf{A}_P^{(k)}} \tag{34}$$

$$\mathbf{HP}_P[\mathbf{u}^{(k)}]=\begin{bmatrix} HP_P[u^{(k)}] \\ HP_P[v^{(k)}] \\ HP_P[w^{(k)}] \end{bmatrix}=\begin{bmatrix} \dfrac{\sum\limits_{NB} A_{NB}^{(k),u}u_{NB}^{(k)}+B_P^{(k),u}+\Omega_P \sum\limits_{m=(phases\neq k)} g^{(km)}u_p^{(m)}}{A_P^{(k),u}} \\ \dfrac{\sum\limits_{NB} A_{NB}^{(k),v}v_{NB}^{(k)}+B_P^{(k),v}+\Omega_P \sum\limits_{m=(phases\neq k)} g^{(km)}v_p^{(m)}}{A_P^{(k),v}} \\ \dfrac{\sum\limits_{NB} A_{NB}^{(k),w}w_{NB}^{(k)}+B_P^{(k),w}+\Omega_P \sum\limits_{m=(phases\neq k)} g^{(km)}w_p^{(m)}}{A_P^{(k),w}} \end{bmatrix} \tag{35}$$

$$\mathbf{HI}_P[\mathbf{u}^{(k)}]=\begin{bmatrix} HI_P[u^{(k)}] \\ HI_P[v^{(k)}] \\ HI_P[w^{(k)}] \end{bmatrix}=\begin{bmatrix} \dfrac{\sum\limits_{NB} A_{NB}^{(k),u}u_{NB}^{(k)}+B_P^{(k),u}-r_P^{(k)}\Omega_P(\nabla P)_P^x}{A_P^{(k),u}} \\ \dfrac{\sum\limits_{NB} A_{NB}^{(k),v}v_{NB}^{(k)}+B_P^{(k),v}-r_P^{(k)}\Omega_P(\nabla P)_P^y}{A_P^{(k),v}} \\ \dfrac{\sum\limits_{NB} A_{NB}^{(k),w}w_{NB}^{(k)}+B_P^{(k),w}-r_P^{(k)}\Omega_P(\nabla P)_P^z}{A_P^{(k),w}} \end{bmatrix} \tag{36}$$

For later use, modified forms of the $\mathbf{HP}_P[\mathbf{u}^{(k)}]$ and $\mathbf{D}_P^{(k)}$ operators are defined as:

$$\tilde{\mathbf{HP}}_P[\mathbf{u}^{(k)}]=\begin{bmatrix} \dfrac{\sum\limits_{NB} A_{NB}^{(k),u}u_{NB}^{(k)}}{A_P^{(k),u}} \\ \dfrac{\sum\limits_{NB} A_{NB}^{(k),v}v_{NB}^{(k)}}{A_P^{(k),v}} \\ \dfrac{\sum\limits_{NB} A_{NB}^{(k),w}w_{NB}^{(k)}}{A_P^{(k),w}} \end{bmatrix} \qquad \tilde{\mathbf{D}}_P^{(k)}=\begin{bmatrix} \dfrac{D_P^{(k)}[u]}{1-\tilde{HP}[1]} & 0 & 0 \\ 0 & \dfrac{D_P^{(k)}[v]}{1-\tilde{HP}[1]} & 0 \\ 0 & 0 & \dfrac{D_P^{(k)}[w]}{1-\tilde{HP}[1]} \end{bmatrix} \tag{37}$$

Discretization of the Continuity Equation

For the volume fraction, density, and velocity fields of the kth phase to satisfy the mass-conservation principle, they have to obey the following differential equation:

$$\frac{\partial\left(r^{(k)}\rho^{(k)}\right)}{\partial t}+\nabla.\left(r^{(k)}\rho^{(k)}\mathbf{u}^{(k)}\right)=r^{(k)}\dot{M}^{(k)} \tag{38}$$

This equation can be viewed as a volume fraction equation for the kth phase in which case it can be discretized and written in the form

$$r_P^{(k)} = H_P\left[r^{(k)}\right] \tag{39}$$

or as a continuity equation for the kth phase, in which case it is discretized in a form to be used for deriving the pressure correction equation:

$$\frac{\left(r_P^{(k)}\rho_P^{(k)}\right)-\left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega - \Delta_P\left[r^{(k)}\mathbf{u}^{(k)}.\mathbf{S}\right] = r^{(k)}\dot{M}^{(k)} \tag{40}$$

where the operator represents the following operation:

$$\Delta_F[\Phi] = \sum_{f=nb(F)=e(F),w(F),n(F),s(F),t(F),b(F)}\left(\Phi_f\right) \tag{41}$$

Discretization of the Energy equation

The discretization of the energy equation follows that of the general multi-fluid scalar equation. The only difference is the one pertaining to the discretization of the additional source terms. Since a control volume approach is followed, the integral of these sources over the control volume appears in the discretized equation. By using the divergence theorem, the volume integral is transformed into surface integral and the resultant discretized expressions evaluated explicitly.

Multi-Fluid Solution Algorithms

Before detailing the multi-fluid segregated solution algorithms, it will be shown that these algorithms may be grouped under two categories denoted in this work by the Geometric Conservation Based Algorithms and the Mass Conservation Based Algorithms. To justify this classification, attention is focussed on the equations and variables involved in a multi-fluid flow situation with n-fluids. In such a case, there will be n momentum equations, n volume

fraction (or mass conservation) equations, a geometric conservation equation, and for the case of a compressible flow an additional n auxiliary pressure-density relations. Moreover, the variables involved are the n velocity vectors, the n volume fractions, the pressure field, and for a compressible flow an additional n unknown density fields. It is clear that the n-velocity fields are associated with the n-momentum equations, i.e. the momentum equations can be directly used to calculate the velocity fields. The volume fractions could arguably be calculated from the volume fraction equations, which means that the remaining equation i.e. the geometric conservation equation (the volume fractions sum to 1) has to be used in deriving the pressure equation, or equivalently the pressure correction equation. This results in what is called the **Geometric Conservation Based Algorithm (GCBA)**.

Alternatively the equations can be arranged differently, with the n momentum equations used to calculate the n velocity fields, n-1 volume fraction (mass conservation) equations used to calculate n-1 volume fraction fields, and the last volume fraction field calculated using the geometric conservation equation i.e.

$$r^{(n)} = 1 - \sum_{all\ k \neq n} r^{(k)} \tag{42}$$

The remaining volume fraction equation can be used to calculate the pressure field. However, instead of using this last volume fraction equation, the global conservation equation can be employed, i.e. the sum of the individual mass conservation equations, to derive the pressure or pressure correction equation. In this case the resulting algorithm is called the **Mass Conservation Based Algorithm (MCBA)**.

With this classification, attention will now be directed towards introducing the various available and possible multi-fluid solution algorithms grouped along the above-mentioned tracks.

Part I: Mass Conservation Based Algorithms

The sequence of events in the Mass Conservation Based Algorithm (MCBA) is as follows:

- Solve the momentum equations for velocities.
- Solve the pressure correction equation based on global mass conservation.
- Correct velocities, densities, and pressure.
- Solve the individual mass conservation equations for volume fractions.
- Solve the energy equations.
- Return to the first step and repeat until convergence.

The above steps, along with some of the techniques that were developed to improve on them, are detailed next.

Solving for Velocities

In this first step, the following systems of momentum equations are solved to find $\mathbf{u}_P^{(k)*}$ based on guessed or previously calculated volume fraction and pressure fields:

$$\mathbf{u}_P^{(k)} = \mathbf{HI}_P\left[\mathbf{u}^{(k)}\right] + \mathbf{D}_P^{(k)} \sum_{m=(phases \neq k)} g^{(km)} \mathbf{u}_P^{(m)} \qquad k = 1....n \tag{43}$$

From Eq. (43) it is clear that the $\mathbf{HI}_P\left[\mathbf{u}^{(k)}\right.$ term couples $\mathbf{u}_P^{(k)}$ to the neighbouring phase (k) velocities (geometric or spatial coupling) while the $\mathbf{D}_P^{(k)} \sum_{m=(phases \neq k)} g^{(km)} \mathbf{u}_P^{(m)}$ term couples $\mathbf{u}_P^{(k)}$ to the velocity of all other phases at grid point P (inter-phase coupling). Therefore, the rate of convergence of the iterative solution procedure used to solve the above system will greatly depend on its capability to resolve both types of coupling. The spatial coupling represents no problem to the well-established iterative techniques since it couples velocities of the same

phase. The inter-phase coupling is however problematic since it relates velocities of different phases. An explicit evaluation of this term slows the convergence rate considerably especially when the inter-fluid momentum transfer terms, represented by $g^{(mn)}$, are large. To accelerate convergence, the Partial Elimination Algorithm (PEA) [68] and the SImultaneous solution of Non-linearly Coupled Equations (SINCE) [67] Technique were developed.

Improvement #1: The Partial Elimination Algorithm and The Simultaneous solution of Non-linearly Coupled Equations Technique

The central idea in the Partial Elimination Algorithm (PEA), applicable for two-fluid flow, is to render the discretized momentum equations more implicit by de-coupling the two sets of equations. This is achieved in a straightforward algebraic manner and it results in a modification to the values of $\mathbf{A}_P^{(k)}$ and $\mathbf{B}_P^{(k)}$.

For the case of two-fluid flow, the momentum equations are given by:

$$\begin{aligned}\mathbf{u}_P^{(1)} &= \mathbf{HI}_P\left[\mathbf{u}^{(1)}\right] + \mathbf{D}_P^{(1)} g^{(12)} \mathbf{u}_P^{(2)} \\ \mathbf{u}_P^{(2)} &= \mathbf{HI}_P\left[\mathbf{u}^{(2)}\right] + \mathbf{D}_P^{(2)} g^{(21)} \mathbf{u}_P^{(1)}\end{aligned} \tag{44}$$

It is clear that each of the equations (44) contains both variables $\mathbf{u}_P^{(1)}$ and $\mathbf{u}_P^{(2)}$ simultaneously. In order to eliminate the velocity of one of the phases from the other phase momentum equation, and vice versa, these equations may be rewritten using the PEA as

$$\begin{aligned}\mathbf{u}_P^{(1)} &= \frac{\mathbf{HI}_P\left[\mathbf{u}^{(1)}\right] + \mathbf{D}_P^{(1)} g^{(12)} \mathbf{HI}_P\left[\mathbf{u}^{(2)}\right]}{1 - \mathbf{D}_P^{(1)} g^{(12)} \mathbf{D}_P^{(2)} g^{(21)}} \\ \mathbf{u}_P^{(2)} &= \frac{\mathbf{HI}_P\left[\mathbf{u}^{(2)}\right] + \mathbf{D}_P^{(2)} g^{(21)} \mathbf{HI}_P\left[\mathbf{u}^{(1)}\right.}{1 - \mathbf{D}_P^{(1)} g^{(12)} \mathbf{D}_P^{(2)} g^{(21)}}\end{aligned} \tag{45}$$

In terms of the coefficients this is equivalent to

$$\mathbf{u}_P^{(1)} = \frac{\mathbf{A}_P^{(2)}\left(\sum\limits_{NB(P)}\mathbf{A}_{NB}^{(1)}\mathbf{u}_{NB}^{(1)} + \mathbf{B}_P^{(1)} - r_P^{(1)}\Omega_P\nabla_P P\right) + \Omega_P g^{(12)}\left(\sum\limits_{NB(P)}\mathbf{A}_{NB}^{(2)}\mathbf{u}_{NB}^{(2)} + \mathbf{B}_P^{(2)} - r_P^{(2)}\Omega_P\nabla_P P\right)}{\mathbf{A}_P^{(1)}\mathbf{A}_P^{(2)} - (\Omega_P)^2 g^{(12)} g^{(21)}}$$
$$\mathbf{u}_P^{(2)} = \frac{\mathbf{A}_P^{(1)}\left(\sum\limits_{NB(P)}\mathbf{A}_{NB}^{(2)}\mathbf{u}_{NB}^{(2)} + \mathbf{B}_P^{(2)} - r_P^{(2)}\Omega_P\nabla_P P\right) + \Omega_P g^{(21)}\left(\sum\limits_{NB(P)}\mathbf{A}_{NB}^{(1)}\mathbf{u}_{NB}^{(1)} + \mathbf{B}_P^{(1)} - r_P^{(1)}\Omega_P\nabla_P P\right)}{\mathbf{A}_P^{(1)}\mathbf{A}_P^{(2)} - (\Omega_P)^2 g^{(12)} g^{(21)}} \tag{46}$$

Equation (46) can be cast in a form similar to equation (25) to yield

$$\overbrace{\left(\frac{\mathbf{A}_P^{(1)}\mathbf{A}_P^{(2)} - (\Omega_P)^2 g^{(12)} g^{(21)}}{\mathbf{A}_P^{(2)}}\right)}^{\tilde{\mathbf{A}}_P^{(1)}} \mathbf{u}_P^{(1)} = \sum_{NB(P)}\left(\mathbf{A}_{NB}^{(1)}\mathbf{u}_{NB}^{(1)}\right) + \overbrace{\frac{\Omega_P g^{(12)}}{\mathbf{A}_P^{(2)}}\left(\sum_{NB(P)}\mathbf{A}_{NB}^{(2)}\mathbf{u}_{NB}^{(2)} + \mathbf{B}_P^{(2)} - r_P^{(2)}\Omega_P\nabla_P P\right) + \left(\mathbf{B}_P^{(1)} - r_P^{(1)}\Omega_P\nabla_P P\right)}^{\tilde{\mathbf{B}}_P^{(1)}}$$
$$\overbrace{\left(\frac{\mathbf{A}_P^{(1)}\mathbf{A}_P^{(2)} - (\Omega_P)^2 g^{(12)} g^{(21)}}{\mathbf{A}_P^{(1)}}\right)}^{\tilde{\mathbf{A}}_P^{(2)}} \mathbf{u}_P^{(2)} = \sum_{NB(P)}\left(\mathbf{A}_{NB}^{(2)}\mathbf{u}_{NB}^{(2)}\right) + \overbrace{\frac{\Omega_P g^{(21)}}{\mathbf{A}_P^{(1)}}\left(\sum_{NB(P)}\mathbf{A}_{NB}^{(1)}\mathbf{u}_{NB}^{(1)} + \mathbf{B}_P^{(1)} - r_P^{(1)}\Omega_P\nabla_P P\right) + \left(\mathbf{B}_P^{(2)} - r_P^{(2)}\Omega_P\nabla_P P\right)}^{\tilde{\mathbf{B}}_P^{(2)}} \tag{47}$$

The use of PEA renders the equations more implicit ($\mathbf{u}_P^{(2)}$ is absent from the $\mathbf{u}_P^{(1)}$ equation and vice versa) and enhances the rate of convergence, which otherwise would have been slowed down by the lagging inter-linkage of the two equations.

It is interesting to note that when g is very large as compared to the $\tilde{\mathbf{A}}_P^{(k)}$ coefficients (i.e. $g^{12} \approx g^{21} = g >> \tilde{\mathbf{A}}_P^{(k)}$) the $\mathbf{u}_P^{(1)}$ and $\mathbf{u}_P^{(2)}$ equations do tend to a common value. For that purpose, the denominator of Eq. (45) is written in terms of the $\tilde{\mathbf{A}}_P^{(k)}$ coefficients and simplified as follows:

$$\begin{aligned}\mathbf{A}_P^{(1)}\mathbf{A}_P^{(2)} - (\Omega_P g)^2 &= \left(\tilde{\mathbf{A}}_P^{(1)} + \Omega_P g\right)\left(\tilde{\mathbf{A}}_P^{(2)} + \Omega_P g\right) - (\Omega_P g)^2 \\ &= \tilde{\mathbf{A}}_P^{(1)}\tilde{\mathbf{A}}_P^{(2)} + \Omega_P g\left(\tilde{\mathbf{A}}_P^{(1)} + \tilde{\mathbf{A}}_P^{(2)}\right)\end{aligned} \tag{48}$$

Using Eq. (48), the $\mathbf{u}_P^{(1)}$ equation reduces to

$$\mathbf{u}_P^{(1)} = \frac{\left(\dfrac{\tilde{\mathbf{A}}_P^{(2)}}{\Omega_P g} + 1\right)\left(\sum\limits_{NB(P)}\mathbf{A}_{NB}^{(1)}\mathbf{u}_{NB}^{(1)} + \mathbf{B}_P^{(1)} - r_P^{(1)}\Omega_P\nabla_P P\right) + \left(\sum\limits_{NB(P)}\mathbf{A}_{NB}^{(2)}\mathbf{u}_{NB}^{(2)} + \mathbf{B}_P^{(2)} - r_P^{(2)}\Omega_P\nabla_P P\right)}{\dfrac{\tilde{\mathbf{A}}_P^{(1)}\tilde{\mathbf{A}}_P^{(2)}}{\Omega_P g} + \left(\tilde{\mathbf{A}}_P^{(1)} + \tilde{\mathbf{A}}_P^{(2)}\right)} \tag{49}$$

In the limit where $g >> A_P^{(k)}$, the equation simplifies to:

$$\mathbf{u}_P^{(1)} = \frac{\left(\sum_{NB(P)} \mathbf{A}_{NB}^{(1)}\mathbf{u}_{NB}^{(1)} + \mathbf{B}_P^{(1)} - r_P^{(1)}\Omega_P \nabla_P P\right) + \left(\sum_{NB(P)} \mathbf{A}_{NB}^{(2)}\mathbf{u}_{NB}^{(2)} + \mathbf{B}_P^{(2)} - r_P^{(2)}\Omega_P \nabla_P P\right)}{\tilde{\mathbf{A}}_P^{(1)} + \tilde{\mathbf{A}}_P^{(2)}} \tag{50}$$

The same expression appears on the right-hand side of the correspondingly reduced equation for $\mathbf{u}_P^{(2)}$. The implication is clear, when the interface mass coefficients are high, $\mathbf{u}_P^{(1)}$ and $\mathbf{u}_P^{(2)}$ are everywhere equal.

The use of the PEA technique with more than two fluids can become cumbersome. The SImultaneous solution of Non-linearly Coupled Equations (SINCE) method [67], is a technique similar to the PEA in its aim, applicable for three or more phases. It is however different than the PEA in that the equations' spatial coupling through the $\mathbf{HI}_P\left[\mathbf{u}^{(k)}\right]$ terms and inter-phase coupling through the $\mathbf{D}_P^{(k)} \sum_{m=all\ fluids \neq k} g^{(km)}\breve{\mathbf{u}}_P^{(m)}$ are accounted for in two distinct steps. In the first step, the inter-phase coupling is resolved by solving the momentum equations on each grid point without accounting for the spatial coupling, i.e. by moving the $\mathbf{HI}_P\left[\mathbf{u}^{(k)}\right]$ terms to the right hand side and treating them as source terms. The equation for a k-fluid flow can be re-arranged and written in the following form:

$$\begin{aligned}
\breve{\mathbf{u}}_P^{(1)} &= \mathbf{D}_P^{(1)} \sum_{m=all\ fluids \neq 1} g^{(1m)}\breve{\mathbf{u}}_P^{(m)} + \mathbf{HI}_P\left[\mathbf{u}^{(1)}\right] \\
\breve{\mathbf{u}}_P^{(2)} &= \mathbf{D}_P^{(2)} \sum_{m=all\ fluids \neq 2} g^{(2m)}\breve{\mathbf{u}}_P^{(m)} + \mathbf{HI}_P\left[\mathbf{u}^{(2)}\right] \\
&\vdots \\
\breve{\mathbf{u}}_P^{(k)} &= \mathbf{D}_P^{(k)} \sum_{m=all\ fluids \neq k} g^{(km)}\breve{\mathbf{u}}_P^{(m)} + \mathbf{HI}_P\left[\mathbf{u}^{(k)}\right]
\end{aligned} \tag{51}$$

These equations are solved using any desired technique (implicit or explicit) for the respective control volumes to give a better estimate of the velocity fields $\left(\breve{\mathbf{u}}_P^{(k)}\right)$ which are then used to calculate the inter-phase friction terms

$$\mathbf{D}_P^{(k)} \sum_{m=all\ fluids \neq k} g^{(km)} \breve{\mathbf{u}}_P^{(m)} \quad for\ \ all\ \ k \tag{52}$$

In the second step, the inter-phase friction terms are absorbed into the standard source terms of equation (25) and the full-field solution of the momentum equations is accomplished in a sequential manner using the normal calculation method via the following system of equations:

$$\mathbf{u}_P^{(k)*} = \mathbf{HI}_P\left[\mathbf{u}^{(k)*}\right] + D_P^{(k)} \sum_{m=all\ \ fluids \neq k} g^{(km)} \breve{\mathbf{u}}_P^{(m)} \quad for\ \ all\ \ k \tag{53}$$

Solving for Pressure Correction

Before convergence, the velocities calculated from the momentum equations do not necessarily satisfy the mass conservation equations. In the segregated approach, the burden of restoring balance rests on the pressure correction equation, which in this case is derived from the overall mass conservation equation. Therefore, the first step in developing a segregated solution algorithm is to derive such an equation. The segregated MCBA can be viewed as an extension to the SIMPLE algorithm and its variants, in which, the pressure equation derivation follow the same pattern as in SIMPLE (or any of its variants), and the corrections are applied only to the velocity, pressure, and density fields. No correction is applied to the volume fractions, rather, they are obtained by solving the individual continuity and geometric constraint equations.

To derive a pressure or pressure-correction equation, the various continuity equations are first added to yield the total mass conservation equation given by:

$$\sum_k \left\{ \frac{\left(r_P^{(k)}\rho_P^{(k)}\right) - \left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega + \Delta_P\left(r^{(k)}\rho_P^{(k)}\mathbf{u}^{(k)}.\mathbf{S}\right) \right\} = \sum_k r^{(k)} \dot{M}^{(k)} = 0 \tag{54}$$

The derivation starts by noticing that in the predictor stage (the previous step) a guessed or estimated pressure field from the previous iteration denoted by P°, is substituted into the momentum equations. The resulting velocity field denoted by $\mathbf{u}^{(k)*}$, which now satisfies the

momentum equations, in general, will not satisfy the continuity equation. Thus, a correction is needed in order to yield velocity and pressure fields that would satisfy both equations. Denoting the corrections for pressure, velocity, and density by P', $\mathbf{u}^{(k)'}$, and $\rho^{(k)'}$ respectively, the corrected fields are written as:

$$\begin{cases} P = P^{\circ} + P' \\ \mathbf{u}^{(k)} = \mathbf{u}^{(k)^*} + \mathbf{u}^{(k)'} \qquad \left(u^{(k)} = u^{(k)^*} + u^{(k)'}, v^{(k)} = v^{(k)^*} + v^{(k)'}, w^{(k)} = w^{(k)^*} + w^{(k)'}\right. \\ \rho^{(k)} = \rho^{(k)^{\circ}} + \rho^{(k)'} \end{cases} \tag{55}$$

Thus, before the pressure field is known, the velocity obtained from the solution of the momentum equations is actually $\mathbf{u}^{(k)*}$ rather than $\mathbf{u}^{(k)}$. Hence the equations solved in the predictor stage are:

$$\mathbf{u}_P^{(k)*} = \mathbf{HP}_P[\mathbf{u}^{(k)^*}] - r^{(k)\circ}\mathbf{D}_P^{(k)}\nabla_P P^{\circ} \tag{56}$$

While the final solution satisfies

$$\mathbf{u}_P^{(k)} = \mathbf{HP}_P[\mathbf{u}^{(k)}] - r^{(k)\circ}\mathbf{D}_P^{(k)}\nabla_P P \tag{57}$$

Subtracting the two sets of equation (57) and (56) from each other yields the following equation involving the correction terms:

$$\mathbf{u}_P^{(k)'} = \mathbf{HP}_P[\mathbf{u}^{(k)'}] - r^{(k)^{\circ}}\mathbf{D}_P^{(k)}\nabla_P P' \tag{58}$$

The velocity and density fields are corrected to satisfy mass conservation. Therefore, the new density and velocity fields, $\rho^{(k)}$ and $\mathbf{u}^{(k)}$, will satisfy the overall mass conservation equation if:

$$\sum_k \left\{ \frac{\left(r_P^{(k)\circ}\rho_P^{(k)}\right) - \left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega + \Delta_P\left[r^{(k)\circ}\rho^{(k)}\mathbf{u}^{(k)}.\mathbf{S}\right] \right\} = 0 \tag{59}$$

Linearizing the $(\rho^{(k)}\mathbf{u}^{(k)})$ term, one gets

$$\left(\rho^{(k)*} + \rho^{(k)'}\right)\left(\mathbf{u}^{(k)*} + \mathbf{u}^{(k)'}\right) = \rho^{(k)*}\mathbf{u}^{(k)*} + \rho^{(k)*}\mathbf{u}^{(k)'} + \rho^{(k)'}\mathbf{u}^{(k)*} + \rho^{(k)'}\mathbf{u}^{(k)'} \tag{60}$$

Substitution of equation (60) into equation (59) gives

$$\sum_k \left\{ \begin{array}{l} \dfrac{\left(r_P^{(k)\circ}\rho_P^{(k)}\right) - \left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega \\ \qquad + \Delta_P\left[r^{(k)\circ}\left(\rho^{(k)*}\mathbf{u}^{(k)*} + \rho^{(k)*}\mathbf{u}^{(k)'} + \rho^{(k)'}\mathbf{u}^{(k)*} + \rho^{(k)'}\mathbf{u}^{(k)'}\right)\mathbf{S}\right] \end{array} \right\} = 0 \tag{61}$$

Rearranging, the following equation is obtained:

$$\begin{aligned} &\sum_k \left\{ \frac{\Omega}{\delta t} r_P^{(k)\circ}\left(\rho_P^{(k)*} + \rho_P^{(k)'}\right) + \Delta_P\left[r^{(k)\circ}\left(\rho^{(k)*}\mathbf{u}^{(k)'} + \rho^{(k)'}\mathbf{u}^{(k)*}\right)\mathbf{S}\right] \right\} \\ &\qquad = \sum_k \left\{ \left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}\frac{\Omega}{\delta t} - \Delta_P\left[r^{(k)\circ}\left(\rho^{(k)*}\mathbf{u}^{(k)*} + \rho^{(k)'}\mathbf{u}^{(k)'}\right)\mathbf{S}\right] \right\} \end{aligned} \tag{62}$$

Using equation (58), the above equation becomes:

$$\begin{aligned} &\sum_k \left\{ \frac{\Omega}{\delta t} r_P^{(k)\circ}\rho_P^{(k)'} + \Delta_P\left[r^{(k)\circ}U^{(k)*}\rho^{(k)'} + r^{(k)\circ}\rho^{(k)*}\left(\mathbf{HP}\left[\mathbf{u}^{(k)'}\right] - r^{(k)^\circ}\mathbf{D}^{(k)}\nabla P'\right)\mathbf{S}\right] \right\} \\ &\qquad = -\sum_k \left\{ \frac{r_P^{(k)\circ}\rho_P^{(k)*} - \left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega + \Delta_P\left[r^{(k)\circ}\rho^{(k)*}\mathbf{u}^{(k)*}.\mathbf{S}\right] + \Delta_P\left[r^{(k)\circ}\rho^{(k)'}\mathbf{u}^{(k)'}.\mathbf{S}\right] \right\} \end{aligned} \tag{63}$$

Finally, substituting density correction by pressure correction, as obtained from the equation of state, the final form of the pressure-correction equation is written as:

$$\begin{aligned} &\sum_k \left\{ \frac{\Omega}{\delta t} r_P^{(k)\circ}C_\rho^{(k)}P_P' + \Delta_P\left[r^{(k)\circ}U^{(k)*}C_\rho^{(k)}P'\right] - \Delta_P\left[r^{(k)^\circ}\rho^{(k)*}\left(r^{(k)^\circ}\mathbf{D}^{(k)}\nabla P'\right)\mathbf{S}\right] \right\} \\ &\qquad = -\sum_k \left\{ \begin{array}{l} \dfrac{r_P^{(k)\circ}\rho_P^{(k)*} - \left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega + \Delta_P\left[r^{(k)\circ}\rho^{(k)*}U^{(k)*}\right] \\ + \Delta_P\left[r^{(k)^\circ}\rho^{(k)*}\left(\mathbf{HP}\left[\mathbf{u}^{(k)'}\right]\right)\mathbf{S}\right] + \Delta_P\left[r^{(k)\circ}\rho^{(k)'}\mathbf{u}^{(k)'}.\mathbf{S}\right] \end{array} \right\} \end{aligned} \tag{64}$$

The second order correction term $\rho^{(k)'}\mathbf{u}^{(k)'}$ is usually neglected. This does not affect neither the convergence rate (i.e. it is considerably smaller than other terms) nor the final solution, since at the state of convergence the correction fields vanish. For this reason, it is neglected in all subsequent derivations. Furthermore, if the $\mathbf{HP}\left[\mathbf{u}^{(k)'}\right.$ term in the above equation is retained, there will result a pressure correction equation relating the pressure correction value at a point to all values in the domain. Eventhough such an equation ensures that the corrected fields will satisfy the continuity and momentum equations, is undesirable because it becomes intractable. To facilitate implementation and reduce cost, simplifying assumptions related to

this term have been introduced. Depending on these assumptions, different algorithms are obtained. These algorithms were originally developed for single-fluid flow and most of them have not yet been extended to multi-fluid flow. It is an objective of this work to perform this extension.

Extending the Segregated class of single-fluid flow algorithms to multi-fluid flow situations

Moukalled and Darwish [21] have recently unified the formulation of the segregated class of algorithms for single-fluid flow. In their work, the SIMPLE [15,16], SIMPLER [16], SIMPLEC [18], SIMPLEST [33], SIMPLEX [20], SIMPLEM [34], PISO [17], and PRIME [72] algorithms were presented in details and the philosophies behind them explained. Therefore, it is sufficient here to present the symbolic form of the multi-fluid version of these algorithms followed by a brief discussion of their merits and the philosophies behind their development.

In the derivations to follow, the superscripts "old" and "o" denote values from the previous time step and values from the previous iteration, respectively. Moreover, the superscripts *, **, ***, and **** represent the first, second, third, and fourth updated values at the current iteration, respectively.

The MCBA following SIMPLE (MCBA-SIMPLE): Symbolic Form

Predictor:

$$\mathbf{u}_P^{(k)*} = \mathbf{HP}_P[\mathbf{u}^{(k)*}] - r^{(k)\circ}\mathbf{D}_P^{(k)}\nabla_P P^{\circ} \tag{65}$$

Corrector:

$$\left(\mathbf{u}^{(k)'}, P', \rho^{(k)'}\right)\left(\mathbf{u}^{(k)**} = \mathbf{u}^{(k)*} + \mathbf{u}^{(k)'}, P^{*} = P^{\circ} + P', \rho^{(k)*} = \rho^{(k)\circ} + \rho^{(k)'}\right. \tag{66}$$

$$\therefore \mathbf{u}_P^{(k)**} = \mathbf{HP}_P[\mathbf{u}^{(k)**}] - r^{(k)\circ}\mathbf{D}_P^{(k)}\nabla_P P^* = \mathbf{HP}_P[\mathbf{u}^{(k)*} + \mathbf{u}^{(k)'}] - r^{(k)\circ}\mathbf{D}_P^{(k)}\nabla_P\left(P^\circ + P'\right) \tag{67}$$

$$\therefore \begin{cases} \mathbf{u}_P^{(k)'} = \mathbf{HP}_P[\mathbf{u}^{(k)'}] - r^{(k)\circ}\mathbf{D}_P^{(k)}\nabla_P P' \\ \rho^{(k)'} = C_\rho^{(k)} P' \end{cases} \tag{68}$$

<u>Condition:</u>

$$\sum_k \left\{ \frac{\left(r_P^{(k)\circ}\rho_P^{(k)*}\right) - \left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega + \Delta_P\left[r^{(k)\circ}\rho^{(k)*}\mathbf{u}^{(k)**}.\mathbf{S}\right] \right\} = 0 \tag{69}$$

$$\therefore \sum_k \left\{ \begin{array}{l} \dfrac{r_P^{(k)\circ}\left(\rho_P^{(k)\circ} + \rho_P^{(k)'}\right) - \left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega \\ \quad + \Delta_P\left[r^{(k)\circ}\left(\rho^{(k)\circ} + \rho^{(k)'}\right)\left(\mathbf{u}^{(k)*} + \mathbf{HP}\left[\mathbf{u}^{(k)'}\right] - r^{(k)\circ}\mathbf{D}^{(k)}\nabla P'\right)\mathbf{S}\right] \end{array} \right\} = 0 \tag{70}$$

$$\therefore \begin{array}{l} \displaystyle\sum_k \left\{ \frac{\Omega}{\delta t} r_P^{(k)\circ} C_\rho^{(k)} P_P' + \Delta_P\left[r^{(k)\circ}C_\rho^{(k)}U^{(k)*}P'\right] - \Delta_P\left[r^{(k)\circ}\rho^{(k)\circ}\left(r^{(k)\circ}\mathbf{D}^{(k)}\nabla P'\right)\mathbf{S}\right] \right\} \\ \displaystyle = -\sum_k \left\{ \begin{array}{l} \dfrac{r_P^{(k)\circ}\rho_P^{(k)\circ} - \left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega + \Delta_P\left[r^{(k)\circ}\rho^{(k)\circ}U^{(k)*}\right] \\ + \Delta_P\left[r^{(k)\circ}\rho^{(k)\circ}\left(\mathbf{HP}\left[\mathbf{u}^{(k)'}\right]\right)\mathbf{S}\right] + \Delta_P\left[r^{(k)\circ}\rho^{(k)'}\mathbf{u}^{(k)'}.\mathbf{S}\right] \end{array} \right\} \end{array} \tag{71}$$

<u>Approximation:</u>

Neglect: $\mathbf{HP}\left[\mathbf{u}^{(k)'}\right.$, $\Delta_P\left[r^{(k)\circ}\rho^{(k)'}\mathbf{u}^{(k)'}.\mathbf{S}\right.$

$$\Rightarrow \mathbf{u}_P^{(k)'} = -r^{(k)\circ}\mathbf{D}_P^{(k)}\nabla_P P' \tag{72}$$

<u>Approximate Equation:</u>

$$\Rightarrow \begin{array}{l} \displaystyle\sum_k \left\{ \frac{\Omega}{\delta t} r_P^{(k)\circ} C_\rho^{(k)} P_P' + \Delta_P\left[r^{(k)\circ}C_\rho^{(k)}U^{(k)*}P'\right] - \Delta_P\left[r^{(k)\circ}\rho^{(k)\circ}\left(r^{(k)\circ}\mathbf{D}^{(k)}\nabla P'\right)\mathbf{S}\right] \right\} \\ \displaystyle = -\sum_k \left\{ \frac{r_P^{(k)\circ}\rho_P^{(k)\circ} - \left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega + \Delta_P\left[r^{(k)\circ}\rho^{(k)\circ}U^{(k)*}\right] \right\} \end{array} \tag{73}$$

A Global MCBA-SIMPLE Iteration

- *Solve implicitly for* $\mathbf{u}^{(k)}$*, using the old pressure and density fields.*
- *Calculate the* $\mathbf{D}^{(k)}$ *fields.*
- *Solve the pressure correction equation.*
- *Correct* $\mathbf{u}^{(k)}$, P *and* $\rho^{(k)}$.
- *Solve implicitly for* $r^{(k)}$.
- *Solve implicitly the energy equations and update the density fields.*
- *Return to the first step and iterate until convergence.*

The MCBA following SIMPLEC (MCBA-SIMPLEC): Symbolic Form

Predictor:

$$\mathbf{u}_P^{(k)*} = \mathbf{HP}_P[\mathbf{u}^{(k)*}] - r^{(k)\circ}\mathbf{D}_P^{(k)}\nabla_P P^\circ \tag{74}$$

Corrector:

$$\left(\mathbf{u}^{(k)'}, P', \rho^{(k)'}\right) \left(\mathbf{u}^{(k)**} = \mathbf{u}^{(k)*} + \mathbf{u}^{(k)'}, P^* = P^\circ + P', \rho^{(k)*} = \rho^{(k)\circ} + \rho^{(k)'}\right. \tag{75}$$

$$\therefore \mathbf{u}_P^{(k)**} = \mathbf{HP}_P[\mathbf{u}^{(k)**}] - r^{(k)\circ}\mathbf{D}_P^{(k)}\nabla_P P^* = \mathbf{HP}_P[\mathbf{u}^{(k)*} + \mathbf{u}^{(k)'}] - r^{(k)\circ}\mathbf{D}_P^{(k)}\nabla_P\left(P^\circ + P'\right) \tag{76}$$

$$\therefore \mathbf{u}_P^{(k)'} = \mathbf{HP}_P[\mathbf{u}^{(k)'}] - r^{(k)\circ}\mathbf{D}_P^{(k)}\nabla_P P' \tag{77}$$

Subtracting $\tilde{\mathbf{HP}}_P[1]\mathbf{u}_P^{(k)'}$ from both sides, one gets

$$\mathbf{u}_P^{(k)'} - \tilde{\mathbf{HP}}_P[1]\mathbf{u}_P^{(k)'} = \mathbf{HP}_P[\mathbf{u}^{(k)'}] - \tilde{\mathbf{HP}}_P[1]\mathbf{u}_P^{(k)'} - r^{(k)\circ}\mathbf{D}_P^{(k)}\nabla_P P' \tag{78}$$

$$\therefore \left(1 - \tilde{\mathbf{HP}}_P[1]\right)\mathbf{u}_P^{(k)'} = \mathbf{HP}_P[\mathbf{u}^{(k)'} - \mathbf{u}_P^{(k)'}] - r^{(k)\circ}\mathbf{D}_P^{(k)}\nabla_P P' \tag{79}$$

$$\therefore \begin{cases} \mathbf{u}_P^{(k)'} = \dfrac{\mathbf{HP}_P[\mathbf{u}^{(k)'} - \mathbf{u}_P^{(k)'}]}{1 - \tilde{\mathbf{HP}}_P[1]} - \dfrac{r^{(k)\circ}\mathbf{D}_P^{(k)}}{1 - \tilde{\mathbf{HP}}_P[1]}\nabla_P P' \\ \rho^{(k)'} = C_\rho^{(k)} P' \end{cases} \tag{80}$$

Condition:

$$\sum_k \left\{ \frac{\left(r_P^{(k)\circ}\rho_P^{(k)*}\right) - \left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega + \Delta_P\left[r^{(k)\circ}\rho^{(k)*}\mathbf{u}^{(k)**}.\mathbf{S}\right] \right\} = 0 \tag{81}$$

$$\therefore \sum_k \left\{ \begin{array}{l} \dfrac{r_P^{(k)\circ}\left(\rho_P^{(k)\circ} + \rho_P^{(k)'}\right) - \left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega \\ + \Delta_P\left[r^{(k)\circ}\left(\rho^{(k)\circ} + \rho^{(k)'}\right)\left(\mathbf{u}^{(k)*} + \dfrac{\mathbf{HP}[\mathbf{u}^{(k)'} - \mathbf{u}_P^{(k)'}]}{1 - \tilde{\mathbf{HP}}[1]} - \dfrac{r^{(k)\circ}\mathbf{D}^{(k)}}{1 - \tilde{\mathbf{HP}}[1]}\nabla P' \right).\mathbf{S} \right] \end{array} \right\} = 0 \tag{82}$$

$$\sum_k \left\{ \frac{\Omega}{\delta t} r_P^{(k)\circ} C_\rho^{(k)} P_P' + \Delta_P \left[r^{(k)\circ} C_\rho^{(k)} U^{(k)*} P' \right] - \Delta_P \left[r^{(k)^\circ} \rho^{(k)\circ} \left(r^{(k)^\circ} \frac{\mathbf{D}^{(k)}}{1-\tilde{\mathbf{HP}}[1]} \nabla P' \right) .\mathbf{S} \right] \right\}$$

$$\therefore = -\sum_k \left\{ \begin{array}{l} \dfrac{r_P^{(k)\circ} \rho_P^{(k)\circ} - \left(r_P^{(k)} \rho_P^{(k)} \right)^{Old}}{\delta t} \Omega + \Delta_P \left[r^{(k)\circ} \rho^{(k)\circ} U^{(k)*} \right] \\ + \Delta_P \left[r^{(k)^\circ} \rho^{(k)\circ} \left(\dfrac{\mathbf{HP}\left[\mathbf{u}^{(k)'} - \mathbf{u}_P^{(k)'} \right]}{1-\tilde{\mathbf{HP}}[1]} \right) .\mathbf{S} \right] + \Delta_P \left[r^{(k)\circ} \rho^{(k)'} \mathbf{u}^{(k)'} .\mathbf{S} \right] \end{array} \right\} \tag{83}$$

Approximation:

Neglect: $\mathbf{HP}\left[\mathbf{u}^{(k)'} - \mathbf{u}_P^{(k)'} \right.$, $\Delta_P \left[r^{(k)\circ} \rho^{(k)'} \mathbf{u}^{(k)'} .\mathbf{S} \right.$

$$\Rightarrow \mathbf{u}_P^{(k)'} = -r^{(k)\circ} \tilde{\mathbf{D}}_P^{(k)} \nabla_P P' \tag{84}$$

Approximate Equation:

$$\sum_k \left\{ \frac{\Omega}{\delta t} r_P^{(k)\circ} C_\rho^{(k)} P_P' + \Delta_P \left[r^{(k)\circ} C_\rho^{(k)} U^{(k)*} P' \right] - \Delta_P \left[r^{(k)^\circ} \rho^{(k)\circ} \left(r^{(k)^\circ} \tilde{\mathbf{D}}^{(k)} \nabla P' \right) \mathbf{S} \right] \right\}$$

$$\Rightarrow = -\sum_k \left\{ \frac{r_P^{(k)\circ} \rho_P^{(k)\circ} - \left(r_P^{(k)} \rho_P^{(k)} \right)^{Old}}{\delta t} \Omega + \Delta_P \left[r^{(k)\circ} \rho^{(k)\circ} U^{(k)*} \right] \right\} \tag{85}$$

A Global MCBA-SIMPLEC Iteration

- *Solve implicitly for u(k), using the old pressure and desity fields.*
- *Calcuate the* $\tilde{\mathbf{D}}^{(k)}$ *fields.*
- *Solve the pressure correction equation.*
- *Correct* $\mathbf{u}^{(k)}$, P *and* $\rho^{(k)}$.
- *Solve implicitly for* $r^{(k)}$.
- *Solve implicitly the energy equations and update the density fields.*
- *Return to the first step and iterate until convergence.*

The MCBA following PRIME (MCBA-PRIME): Symbolic Form

Predictor:

$$\mathbf{u}_P^{(k)*} = \mathbf{HP}_P[\mathbf{u}^{(k)\circ}] - r^{(k)\circ} \mathbf{D}_P^{(k)} \nabla_P P^\circ \tag{86}$$

Corrector:

$$\left(\mathbf{u}^{(k)'}, P', \rho^{(k)'} \right) \left(\mathbf{u}^{(k)**} = \mathbf{u}^{(k)*} + \mathbf{u}^{(k)'}, P^* = P^\circ + P', \rho^{(k)*} = \rho^{(k)\circ} + \rho^{(k)'} \right. \tag{87}$$

$$\therefore \mathbf{u}_P^{(k)**} = \mathbf{HP}_P[\mathbf{u}^{(k)**}] - r^{(k)\circ} \mathbf{D}_P^{(k)} \nabla_P P^* = \mathbf{HP}_P[\mathbf{u}^{(k)*} + \mathbf{u}^{(k)'}] - r^{(k)\circ} \mathbf{D}_P^{(k)} \nabla_P \left(P^\circ + P' \right) \tag{88}$$

$$\therefore \begin{cases} \mathbf{u}_P^{(k)'} = \mathbf{HP}_p[\mathbf{u}^{(k)*} - \mathbf{u}^{(k)\circ}] + \mathbf{HP}_p[\mathbf{u}^{(k)'}] - r^{(k)\circ}\mathbf{D}_P^{(k)}\nabla_P P' \\ \rho^{(k)'} = C_\rho^{(k)} P' \end{cases} \tag{89}$$

Condition:

$$\sum_k \left\{ \frac{\left(r_P^{(k)\circ}\rho_P^{(k)*}\right) - \left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega + \Delta_P\left[r^{(k)\circ}\rho^{(k)*}\mathbf{u}^{(k)**}.\mathbf{S}\right] \right\} = 0 \tag{90}$$

$$\therefore \sum_k \left\{ \begin{aligned} &\frac{r_P^{(k)\circ}\left(\rho_P^{(k)\circ} + \rho_P^{(k)'}\right) - \left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega \\ &+ \Delta_P\left[r^{(k)\circ}\left(\rho^{(k)\circ} + \rho^{(k)'}\right)\left(\mathbf{u}^{(k)*} + \mathbf{HP}\left[\mathbf{u}^{(k)*} - \mathbf{u}^{(k)\circ}\right] + \mathbf{HP}\left[\mathbf{u}^{(k)'}\right] - r^{(k)\circ}\mathbf{D}^{(k)}\nabla P'\right)\mathbf{S}\right] \end{aligned} \right\} = 0 \tag{91}$$

$$\therefore \begin{aligned} &\sum_k \left\{ \frac{\Omega}{\delta t} r_P^{(k)\circ} C_\rho^{(k)} P_P' + \Delta_P\left[r^{(k)\circ} C_\rho^{(k)} U^{(k)*} P'\right] - \Delta_P\left[r^{(k)\circ}\rho^{(k)\circ}\left(r^{(k)\circ}\mathbf{D}^{(k)}\nabla P'\right)\mathbf{S}\right] \right\} \\ &= -\sum_k \left\{ \begin{aligned} &\frac{r_P^{(k)\circ}\rho_P^{(k)\circ} - \left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega + \Delta_P\left[r^{(k)\circ}\rho^{(k)\circ}U^{(k)*}\right] \\ &+ \Delta_P\left[r^{(k)\circ}\rho^{(k)\circ}\left(\mathbf{HP}\left[\mathbf{u}^{(k)*} - \mathbf{u}^{(k)\circ}\right] + \mathbf{HP}\left[\mathbf{u}^{(k)'}\right]\right)\mathbf{S}\right] + \Delta_P\left[r^{(k)\circ}\rho^{(k)'}\mathbf{u}^{(k)'}.\mathbf{S}\right] \end{aligned} \right\} \end{aligned} \tag{92}$$

Approximation:

Neglect: $\mathbf{HP}\left[\mathbf{u}^{(k)*} - \mathbf{u}^{(k)\circ}\right.$, $\mathbf{HP}\left[\mathbf{u}^{(k)'}\right.$, $\Delta_P\left[r^{(k)\circ}\rho^{(k)'}\mathbf{u}^{(k)'}.\mathbf{S}\right.$

Approximate Equation:

$$\Rightarrow \begin{aligned} &\sum_k \left\{ \frac{\Omega}{\delta t} r_P^{(k)\circ} C_\rho^{(k)} P_P' + \Delta_P\left[r^{(k)\circ} C_\rho^{(k)} U^{(k)*} P'\right] - \Delta_P\left[r^{(k)\circ}\rho^{(k)\circ}\left(r^{(k)\circ}\mathbf{D}^{(k)}\nabla P'\right)\mathbf{S}\right] \right\} \\ &= -\sum_k \left\{ \frac{r_P^{(k)\circ}\rho_P^{(k)\circ} - \left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega + \Delta_P\left[r^{(k)\circ}\rho^{(k)\circ}U^{(k)*}\right] \right\} \end{aligned} \tag{93}$$

A Global MCBA-PRIME Iteration

- *Solve explicitly for* $\mathbf{u}^{(k)}$, *using the old pressure and density fields.*
- *Calculate the* $\mathbf{D}^{(k)}$ *fields.*
- *Solve the pressure correction equation.*
- *Correct* $\mathbf{u}^{(k)}$, P *and* $\rho^{(k)}$.
- *Solve implicitly for* $r^{(k)}$.
- *Solve implicitly the energy equations and update the density fields.*
- *Return to the first step and iterate until convergence.*

The MCBA following SIMPLEST (MCBA-SIMPLEST): Symbolic Form

Predictor:

$$\mathbf{u}_P^{(k)*} = \mathbf{HP}_P^D\left[\mathbf{u}^{(k)*}\right] + \mathbf{HP}_P^C\left[\mathbf{u}^{(k)\circ}\right] - r^{(k)\circ}\mathbf{D}_P^{(k)*}\nabla_P P^\circ \tag{94}$$

Corrector:

$$\left(\mathbf{u}^{(k)'}, P', \rho^{(k)'}\right)\left(\mathbf{u}^{(k)**} = \mathbf{u}^{(k)*} + \mathbf{u}^{(k)'}, P^* = P^\circ + P', \rho^{(k)*} = \rho^{(k)\circ} + \rho^{(k)'}\right) \tag{95}$$

$$\begin{aligned} \mathbf{u}_P^{(k)**} &= \mathbf{HP}_P^D\left[\mathbf{u}^{(k)**}\right] + \mathbf{HP}_P^C\left[\mathbf{u}^{(k)**}\right] - r^{(k)\circ}\mathbf{D}_P^{(k)}\nabla_P P^* \\ \therefore \quad &= \mathbf{HP}_P^D\left[\mathbf{u}^{(k)*} + \mathbf{u}^{(k)'}\right] + \mathbf{HP}_P^C\left[\mathbf{u}^{(k)*} + \mathbf{u}^{(k)'}\right] - r^{(k)\circ}\mathbf{D}_P^{(k)}\nabla_P\left(P^\circ + P'\right) \\ &= \mathbf{HP}_P^D\left[\mathbf{u}^{(k)*}\right] + \mathbf{HP}_P^D\left[\mathbf{u}^{(k)'}\right] + \mathbf{HP}_P^C\left[\mathbf{u}^{(k)*}\right] + \mathbf{HP}_P^C\left[\mathbf{u}^{(k)'}\right] \\ &\qquad - r^{(k)\circ}\mathbf{D}_P^{(k)}\nabla_P P^\circ - r^{(k)\circ}\mathbf{D}_P^{(k)}\nabla_P P' \end{aligned} \tag{96}$$

$$\therefore \begin{cases} \mathbf{u}_P^{(k)'} = \mathbf{HP}_P[\mathbf{u}^{(k)'}] + \mathbf{HP}_P^C[\mathbf{u}^{(k)*} - \mathbf{u}^{(k)\circ}] - r^{(k)\circ}\mathbf{D}_P^{(k)}\nabla_P P' \\ \rho^{(k)'} = C_\rho^{(k)} P' \end{cases} \tag{97}$$

Condition:

$$\sum_k \left\{ \frac{\left(r_P^{(k)\circ}\rho_P^{(k)*}\right) - \left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega + \Delta_P\left[r^{(k)\circ}\rho^{(k)*}\mathbf{u}^{(k)**}.\mathbf{S}\right] \right\} = 0 \tag{98}$$

$$\therefore \sum_k \left\{ \begin{aligned} &\frac{r_P^{(k)\circ}\left(\rho_P^{(k)\circ} + \rho_P^{(k)'}\right) - \left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega \\ &+ \Delta_P\left[r^{(k)\circ}\left(\rho^{(k)\circ} + \rho^{(k)'}\right) \begin{pmatrix} \mathbf{u}^{(k)*} + \mathbf{HP}^C\left[\mathbf{u}^{(k)*} - \mathbf{u}^{(k)\circ}\right] \\ + \mathbf{HP}\left[\mathbf{u}^{(k)'}\right] - r^{(k)\circ}\mathbf{D}^{(k)}\nabla P' \end{pmatrix} .\mathbf{S} \right] \end{aligned} \right\} = 0 \tag{99}$$

$$\begin{aligned} &\sum_k \left\{ \frac{\Omega}{\delta t} r_P^{(k)\circ} C_\rho^{(k)} P_P' + \Delta_P\left[r^{(k)\circ}C_\rho^{(k)}U^{(k)*}P'\right] - \Delta_P\left[r^{(k)^\circ}\rho^{(k)\circ}\left(r^{(k)^\circ}\mathbf{D}^{(k)}\nabla P'\right)\mathbf{S}\right] \right\} \\ \therefore \quad &= -\sum_k \left\{ \begin{aligned} &\frac{r_P^{(k)\circ}\rho_P^{(k)\circ} - \left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega + \Delta_P\left[r^{(k)\circ}\rho^{(k)\circ}U^{(k)*}\right] \\ &+ \Delta_P\left[r^{(k)^\circ}\rho^{(k)\circ}\left(\mathbf{HP}^C\left[\mathbf{u}^{(k)*} - \mathbf{u}^{(k)\circ}\right] + \mathbf{HP}\left[\mathbf{u}^{(k)'}\right]\right)\mathbf{S}\right] + \Delta_P\left[r^{(k)\circ}\rho^{(k)'}\mathbf{u}^{(k)'}.\mathbf{S}\right] \end{aligned} \right\} \end{aligned} \tag{100}$$

Approximation:

Neglect: $\mathbf{HP}^C\left[\mathbf{u}^{(k)*} - \mathbf{u}^{(k)\circ}\right.$, $\mathbf{HP}\left[\mathbf{u}^{(k)'}\right.$, $\Delta_P\left[r^{(k)\circ}\rho^{(k)'}\mathbf{u}^{(k)'}.\mathbf{S}\right.$

$$\Rightarrow \mathbf{u}_P^{(k)'} = -r^{(k)\circ}\mathbf{D}_P^{(k)}\nabla_P P' \tag{101}$$

Approximate Equation:

$$\sum_k \left\{ \frac{\Omega}{\delta t} r_P^{(k)\circ} C_\rho^{(k)} P'_P + \Delta_P \left[r^{(k)\circ} C_\rho^{(k)} U^{(k)*} P' \right] - \Delta_P \left[r^{(k)\circ} \rho^{(k)\circ} \left(r^{(k)\circ} \mathbf{D}^{(k)} \nabla P' \right) \mathbf{S} \right] \right\}$$
$$\Rightarrow \quad = -\sum_k \left\{ \frac{r_P^{(k)\circ} \rho_P^{(k)\circ} - \left(r_P^{(k)} \rho_P^{(k)} \right)^{Old}}{\delta t} \Omega + \Delta_P \left[r^{(k)\circ} \rho^{(k)\circ} U^{(k)*} \right] \right\} \tag{102}$$

A Global MCBA-SIMPLEST Iteration

- *Solve for* $\mathbf{u}^{(k)}$, *treating* $\mathbf{HP}^D\left[\mathbf{u}^{(k)}\right.$ *implicitly and* $\mathbf{HP}^C\left[\mathbf{u}^{(k)}\right.$ *explicitly.*
- *Calculate the* $\mathbf{D}^{(k)}$ *fields.*
- *Solve the pressure correction equation.*
- *Correct* $\mathbf{u}^{(k)}$, P *and* $\rho^{(k)}$.
- *Solve implicitly for* $r^{(k)}$.
- *Solve implicitly the energy equations and update the density fields.*
- *Return to the first step and iterate until convergence.*

The MCBA following SIMPLER (MCBA-SIMPLER): Symbolic Form

First Predictor:

No predictor stage. Only coefficients of the momentum equations are calculated.

First Corrector:

$$\left(\mathbf{u}^{(k)'}, P', \rho^{(k)'} \right) \left(\mathbf{u}^{(k)*} = \mathbf{u}^{(k)\circ} + \mathbf{u}^{(k)'}, P^* = P^\circ + P', \rho^{(k)*} = \rho^{(k)\circ} + \rho^{(k)'} \right. \tag{103}$$

$$\therefore \mathbf{u}_P^{(k)*} = \mathbf{HP}_P\left[\mathbf{u}^{(k)*}\right] - r^{(k)\circ} \mathbf{D}_P^{(k)} \nabla_P P^* \tag{104}$$

$$\mathbf{u}_P^{(k)\circ} = \mathbf{HP}_P\left[\mathbf{u}^{(k)\circ}\right] - r^{(k)\circ} \mathbf{D}_P^{(k)} \nabla_P P^\circ \tag{105}$$

$$\therefore \begin{cases} \mathbf{u}_P^{(k)'} = \mathbf{HP}_P[\mathbf{u}^{(k)'}] - r^{(k)\circ} \mathbf{D}_P^{(k)} \nabla_P P' \\ \rho^{(k)'} = C_\rho^{(k)} P' \end{cases} \tag{106}$$

Condition:

$$\sum_k \left\{ \frac{\left(r_P^{(k)\circ} \rho_P^{(k)*} \right) - \left(r_P^{(k)} \rho_P^{(k)} \right)^{Old}}{\delta t} \Omega + \Delta_P \left[r^{(k)\circ} \rho^{(k)*} \mathbf{u}^{(k)*} . \mathbf{S} \right] \right\} = 0 \tag{107}$$

$$\therefore \sum_k \left\{ \begin{array}{l} \dfrac{r_P^{(k)\circ} \left(\rho_P^{(k)\circ} + \rho_P^{(k)'} \right) - \left(r_P^{(k)} \rho_P^{(k)} \right)^{Old}}{\delta t} \Omega \\ + \Delta_P \left[r^{(k)\circ} \left(\rho^{(k)\circ} + \rho^{(k)'} \right) \left(\mathbf{u}^{(k)\circ} + \mathbf{HP}\left[\mathbf{u}^{(k)'}\right] - r^{(k)\circ} \mathbf{D}^{(k)} \nabla P' \right) \mathbf{S} \right] \end{array} \right\} = 0 \tag{108}$$

$$\therefore \sum_k \left\{ \frac{\Omega}{\delta t} r_P^{(k)\circ} C_\rho^{(k)} P_P' + \Delta_P\left[r^{(k)\circ} C_\rho^{(k)} U^{(k)\circ} P'\right] - \Delta_P\left[r^{(k)\circ} \rho^{(k)\circ}\left(r^{(k)\circ}\mathbf{D}^{(k)}\nabla P'\right)\mathbf{S}\right] \right\} = -\sum_k \left\{ \begin{array}{l} \dfrac{r_P^{(k)\circ}\rho_P^{(k)\circ} - \left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega + \Delta_P\left[r^{(k)\circ}\rho^{(k)\circ}U^{(k)\circ}\right] \\ + \Delta_P\left[r^{(k)\circ}\rho^{(k)\circ}\left(\mathbf{HP}\left[\mathbf{u}^{(k)'}\right]\right)\mathbf{S}\right] + \Delta_P\left[r^{(k)\circ}\rho^{(k)'}\mathbf{u}^{(k)'}.\mathbf{S}\right] \end{array} \right\} \quad (109)$$

<u>Approximation:</u>

Neglect: $\mathbf{HP}\left[\mathbf{u}^{(k)'}\right.$, $\Delta_P\left[r^{(k)\circ}\rho^{(k)'}\mathbf{u}^{(k)'}.\mathbf{S}\right.$

$$\Rightarrow \mathbf{u}_P^{(k)'} = -r^{(k)\circ}\mathbf{D}_P^{(k)}\nabla_P P' \quad (110)$$

<u>Approximate Equation:</u>

$$\Rightarrow \sum_k \left\{ \frac{\Omega}{\delta t} r_P^{(k)\circ} C_\rho^{(k)} P_P' + \Delta_P\left[r^{(k)\circ} C_\rho^{(k)} U^{(k)\circ} P'\right] - \Delta_P\left[r^{(k)\circ} \rho^{(k)\circ}\left(r^{(k)\circ}\mathbf{D}^{(k)}\nabla P'\right)\mathbf{S}\right] \right\} = -\sum_k \left\{ \frac{r_P^{(k)\circ}\rho_P^{(k)\circ} - \left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega + \Delta_P\left[r^{(k)\circ}\rho^{(k)\circ}U^{(k)\circ}\right] \right\} \quad (111)$$

Apply correction to pressure and density fields only.

<u>Second Predictor:</u>

$$\mathbf{u}_P^{(k)*} = \mathbf{HP}_P[\mathbf{u}^{(k)*}] - r^{(k)\circ}\mathbf{D}_P^{(k)}\nabla_P P^* \quad (112)$$

<u>Second Corrector:</u>

$$\left(\mathbf{u}^{(k)''}, P'', \rho^{(k)''}\right)\left(\mathbf{u}^{(k)**} = \mathbf{u}^{(k)*} + \mathbf{u}^{(k)''}, P^{**} = P^* + P'', \rho^{(k)**} = \rho^{(k)*} + \rho^{(k)''}\right. \quad (113)$$

$$\therefore \mathbf{u}_P^{(k)**} = \mathbf{HP}_P\left[\mathbf{u}^{(k)**}\right] - r^{(k)\circ}\mathbf{D}_P^{(k)}\nabla_P P^{**} = \mathbf{HP}_P\left[\mathbf{u}^{(k)*} + \mathbf{u}^{(k)''}\right] - r^{(k)\circ}\mathbf{D}_P^{(k)}\nabla_P\left(P^* + P''\right. \quad (114)$$

$$\therefore \left\{ \begin{array}{l} \mathbf{u}_P^{(k)''} = \mathbf{HP}_P[\mathbf{u}^{(k)''}] - r^{(k)\circ}\mathbf{D}_P^{(k)}\nabla_P P'' \\ \rho^{(k)''} = C_\rho^{(k)} P'' \end{array} \right. \quad (115)$$

<u>Condition:</u>

$$\sum_k \left\{ \frac{\left(r_P^{(k)\circ}\rho_P^{(k)**}\right) - \left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega + \Delta_P\left[r^{(k)\circ}\rho^{(k)**}\mathbf{u}^{(k)**}.\mathbf{S}\right] \right\} = 0 \quad (116)$$

$$\therefore \sum_k \left\{ \begin{array}{l} \dfrac{r_P^{(k)\circ}\left(\rho_P^{(k)*} + \rho_P^{(k)''}\right) - \left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega \\ + \Delta_P\left[r^{(k)\circ}\left(\rho^{(k)*} + \rho^{(k)''}\right)\left(\mathbf{u}^{(k)*} + \mathbf{HP}\left[\mathbf{u}^{(k)''}\right] - r^{(k)\circ}\mathbf{D}^{(k)}\nabla P''\right)\mathbf{S}\right] \end{array} \right\} = 0 \quad (117)$$

$$\sum_k \left\{ \frac{\Omega}{\delta t} r_P^{(k)\circ} C_\rho^{(k)} P_P' + \Delta_P \left[r^{(k)\circ} C_\rho^{(k)} U^{(k)*} P' \right] - \Delta_P \left[r^{(k)\circ} \rho^{(k)*} \left(r^{(k)\circ} \mathbf{D}^{(k)} \nabla P'' \right) \mathbf{S} \right] \right\}$$

$$\therefore \quad = -\sum_k \left\{ \begin{array}{l} \dfrac{r_P^{(k)\circ} \rho_P^{(k)*} - \left(r_P^{(k)} \rho_P^{(k)} \right)^{Old}}{\delta t} \Omega + \Delta_P \left[r^{(k)\circ} \rho^{(k)*} U^{(k)*} \right] \\ + \Delta_P \left[r^{(k)\circ} \rho^{(k)*} \left(\mathbf{HP}\left[\mathbf{u}^{(k)''} \right] \right) \mathbf{S} \right] + \Delta_P \left[r^{(k)\circ} \rho^{(k)''} \mathbf{u}^{(k)''} . \mathbf{S} \right] \end{array} \right\} \tag{118}$$

<u>Approximation:</u>

Neglect: $\mathbf{HP}_P\left[\mathbf{u}^{(k)''}\right.$, $\Delta_P\left[r^{(k)\circ}\rho^{(k)''}\mathbf{u}^{(k)''}.\mathbf{S}\right.$

$$\Rightarrow \mathbf{u}_P^{(k)''} = -r^{(k)\circ} \mathbf{D}_P^{(k)} \nabla_P P'' \tag{119}$$

<u>Approximate Equation:</u>

$$\sum_k \left\{ \frac{\Omega}{\delta t} r_P^{(k)\circ} C_\rho^{(k)} P_P' + \Delta_P \left[r^{(k)\circ} C_\rho^{(k)} U^{(k)*} P' \right] - \Delta_P \left[r^{(k)\circ} \rho^{(k)*} \left(r^{(k)\circ} \mathbf{D}^{(k)} \nabla P'' \right) \mathbf{S} \right] \right\}$$

$$\Rightarrow \quad = -\sum_k \left\{ \frac{r_P^{(k)\circ} \rho_P^{(k)*} - \left(r_P^{(k)} \rho_P^{(k)} \right)^{Old}}{\delta t} \Omega + \Delta_P \left[r^{(k)\circ} \rho^{(k)*} U^{(k)*} \right] \right\} \tag{120}$$

Do not correct pressure.

A Global MCBA-SIMPLER Iteration

- *Calculate the $\mathbf{D}^{(k)}$ fields.*
- *Solve the pressure correction equation and update the pressure and density fields.*
- *Solve implicitly for $\mathbf{u}^{(k)}$ using the new pressure and density fields*
- *Solve the pressure correction equation using the new velocity fields to obtain a new pressure correction field.*
- *Correct $\mathbf{u}^{(k)}$.*
- *Solve implicitly for $r^{(k)}$.*
- *Solve implicitly the energy equations and update the density fields.*
- *Return to the first step and iterate until convergence.*

The MCBA following PISO (MCBA-PISO): Symbolic Form

First Predictor:

$$\mathbf{u}_P^{(k)*} = \mathbf{HP}_P[\mathbf{u}^{(k)*}] - r^{(k)\circ}\mathbf{D}_P^{(k)}\nabla_P P^\circ \tag{121}$$

First Corrector:

$$\left(\mathbf{u}^{(k)'}, P', \rho^{(k)'}\right)\left(\mathbf{u}^{(k)**} = \mathbf{u}^{(k)*} + \mathbf{u}^{(k)'}, P^* = P^\circ + P', \rho^{(k)*} = \rho^{(k)\circ} + \rho^{(k)'}\right. \tag{122}$$

$$\therefore \mathbf{u}_P^{(k)**} = \mathbf{HP}_P[\mathbf{u}^{(k)**}] - r^{(k)\circ}\mathbf{D}_P^{(k)}\nabla_P P^* = \mathbf{HP}_P[\mathbf{u}^{(k)*} + \mathbf{u}^{(k)'}] - r^{(k)\circ}\mathbf{D}_P^{(k)}\nabla_P\left(P^\circ + P'\right. \tag{123}$$

$$\therefore \begin{cases} \mathbf{u}_P^{(k)'} = \mathbf{HP}_P[\mathbf{u}^{(k)'}] - r^{(k)\circ}\mathbf{D}_P^{(k)}\nabla_P P' \\ \rho^{(k)'} = C_\rho^{(k)} P' \end{cases} \tag{124}$$

Condition:

$$\sum_k \left\{ \frac{\left(r_P^{(k)\circ}\rho_P^{(k)*}\right) - \left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega + \Delta_P\left[r^{(k)\circ}\rho^{(k)*}\mathbf{u}^{(k)**}.\mathbf{S}\right] \right\} = 0 \tag{125}$$

$$\therefore \sum_k \left\{ \begin{array}{l} \dfrac{r_P^{(k)\circ}\left(\rho_P^{(k)\circ} + \rho_P^{(k)'}\right) - \left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega \\ \quad + \Delta_P\left[r^{(k)\circ}\left(\rho^{(k)\circ} + \rho^{(k)'}\right)\left(\mathbf{u}^{(k)*} + \mathbf{HP}\left[\mathbf{u}^{(k)'}\right] - r^{(k)\circ}\mathbf{D}^{(k)}\nabla P'\right)\mathbf{S}\right] \end{array} \right\} = 0 \tag{126}$$

$$\therefore \begin{array}{l} \displaystyle\sum_k \left\{ \frac{\Omega}{\delta t} r_P^{(k)\circ} C_\rho^{(k)} P_P' + \Delta_P\left[r^{(k)\circ} C_\rho^{(k)} U^{(k)*} P'\right] - \Delta_P\left[r^{(k)\circ}\rho^{(k)\circ}\left(r^{(k)\circ}\mathbf{D}^{(k)}\nabla P'\right)\mathbf{S}\right] \right\} \\ \qquad = -\displaystyle\sum_k \left\{ \begin{array}{l} \dfrac{r_P^{(k)\circ}\rho_P^{(k)\circ} - \left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega + \Delta_P\left[r^{(k)\circ}\rho^{(k)\circ}U^{(k)*}\right] \\ + \Delta_P\left[r^{(k)\circ}\rho^{(k)\circ}\left(\mathbf{HP}\left[\mathbf{u}^{(k)'}\right]\right)\mathbf{S}\right] + \Delta_P\left[r^{(k)\circ}\rho^{(k)'}\mathbf{u}^{(k)'}.\mathbf{S}\right] \end{array} \right\} \end{array} \tag{127}$$

Approximation:

Neglect: $\mathbf{HP}\left[\mathbf{u}^{(k)'}\right.$, $\Delta_P\left[r^{(k)\circ}\rho^{(k)'}\mathbf{u}^{(k)'}.\mathbf{S}\right.$

$$\Rightarrow \mathbf{u}_P^{(k)'} = -r^{(k)\circ}\mathbf{D}_P^{(k)}\nabla_P P' \tag{128}$$

Approximate Equation:

$$\begin{aligned}&\sum_k\left\{\frac{\Omega}{\delta t}r_P^{(k)\circ}C_\rho^{(k)}P'_P+\Delta_P\left[r^{(k)\circ}C_\rho^{(k)}U^{(k)*}P'\right]-\Delta_P\left[r^{(k)\circ}\rho^{(k)\circ}\left(r^{(k)\circ}\mathbf{D}^{(k)}\nabla P'\right)\mathbf{S}\right]\right\}\\ \Rightarrow\;&\qquad=-\sum_k\left\{\frac{r_P^{(k)\circ}\rho_P^{(k)\circ}-\left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega+\Delta_P\left[r^{(k)\circ}\rho^{(k)\circ}U^{(k)*}\right]\right\}\end{aligned} \tag{129}$$

Second Corrector:

$$\left(\mathbf{u}^{(k)''},P'',\rho^{(k)''}\right)\left(\mathbf{u}^{(k)****}=\mathbf{u}^{(k)***}+\mathbf{u}^{(k)''},P^{**}=P^{*}+P'',\rho^{(k)**}=\rho^{(k)*}+\rho^{(k)''}\right. \tag{131}$$

$$\therefore\mathbf{u}_P^{(k)****}=\mathbf{HP}_P^{**}[\mathbf{u}^{(k)****}]-r^{(k)\circ}\mathbf{D}_P^{(k)**}\nabla_P\left(P^{*}+P''\right) \tag{132}$$

$$\mathbf{u}_P^{(k)***}=\mathbf{HP}_P^{**}[\mathbf{u}^{(k)**}]-r^{(k)\circ}\mathbf{D}_P^{(k)**}\nabla_P P^{*} \tag{133}$$

$$\therefore\begin{cases}\mathbf{u}_P^{(k)''}=\mathbf{HP}_P^{**}[\mathbf{u}^{(k)****}-\mathbf{u}^{(k)**}]-r^{(k)\circ}\mathbf{D}_P^{(k)**}\nabla_P P''\\ \qquad=\mathbf{HP}_P^{**}[\mathbf{u}^{(k)***}-\mathbf{u}^{(k)**}+\mathbf{u}^{(k)''}]-r^{(k)\circ}\mathbf{D}_P^{(k)**}\nabla_P P''\\ \rho^{(k)''}=C_\rho^{(k)}P''\end{cases} \tag{134}$$

Condition:

$$\sum_k\left\{\frac{\left(r_P^{(k)\circ}\rho_P^{(k)**}\right)-\left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega+\Delta_P\left[r^{(k)\circ}\rho^{(k)**}\mathbf{u}^{(k)****}.\mathbf{S}\right]\right\}=0 \tag{135}$$

$$\therefore\sum_k\left\{\begin{aligned}&\frac{r_P^{(k)\circ}\left(\rho_P^{(k)*}+\rho_P^{(k)''}\right)-\left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega\\ &+\Delta_P\left[r^{(k)\circ}\left(\rho^{(k)*}+\rho^{(k)''}\right)\left(\begin{aligned}&\mathbf{u}^{(k)***}+\mathbf{HP}^{**}\left[\mathbf{u}^{(k)***}-\mathbf{u}^{(k)**}+\mathbf{u}^{(k)''}\right]\\&-r^{(k)\circ}\mathbf{D}^{(k)**}\nabla P'\end{aligned}\right).\mathbf{S}\right]\end{aligned}\right\}=0 \tag{136}$$

$$\begin{aligned}&\sum_k\left\{\frac{\Omega}{\delta t}r_P^{(k)\circ}C_\rho^{(k)}P''_P+\Delta_P\left[r^{(k)\circ}C_\rho^{(k)}U^{(k)***}P''\right]-\Delta_P\left[r^{(k)\circ}\rho^{(k)*}\left(r^{(k)\circ}\mathbf{D}^{(k)**}\nabla P''\right)\mathbf{S}\right]\right\}\\ \therefore\;&=-\sum_k\left\{\begin{aligned}&\frac{r_P^{(k)\circ}\rho_P^{(k)*}-\left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega+\Delta_P\left[r^{(k)\circ}\rho^{(k)*}U^{(k)***}\right]\\ &+\Delta_P\left[r^{(k)\circ}\rho^{(k)*}\left(\mathbf{HP}^{**}\left[\mathbf{u}^{(k)***}-\mathbf{u}^{(k)**}+\mathbf{u}^{(k)''}\right]\right)\mathbf{S}\right]+\Delta_P\left[r^{(k)\circ}\rho^{(k)''}\mathbf{u}^{(k)''}.\mathbf{S}\right]\end{aligned}\right\}\end{aligned} \tag{137}$$

Approximation:

Neglect: $\mathbf{HP}^{**}\left[\mathbf{u}^{(k)***}-\mathbf{u}^{(k)**}+\mathbf{u}^{(k)''}\right.$, $\Delta_P\left[r^{(k)\circ}\rho^{(k)''}\mathbf{u}^{(k)''}.\mathbf{S}\right.$

$$\Rightarrow\mathbf{u}_P^{(k)''}=-r^{(k)\circ}\mathbf{D}_P^{(k)**}\nabla_P P'' \tag{138}$$

Approximate Equation:

$$\sum_k \left\{ \frac{\Omega}{\delta t} r_P^{(k)\circ} C_\rho^{(k)} P_P'' + \Delta_P \left[r^{(k)\circ} C_\rho^{(k)} U^{(k)***} P'' \right] - \Delta_P \left[r^{(k)\circ} \rho^{(k)*} \left(r^{(k)\circ} \mathbf{D}^{(k)**} \nabla P'' \right) \mathbf{S} \right] \right\}$$
$$= -\sum_k \left\{ \frac{r_P^{(k)\circ} \rho_P^{(k)*} - \left(r_P^{(k)} \rho_P^{(k)} \right)^{Old}}{\delta t} \Omega + \Delta_P \left[r^{(k)\circ} \rho^{(k)*} U^{(k)***} \right] \right\} \tag{139}$$

A Global MCBA-PISO Iteration

- *Solve implicitly for* $\mathbf{u}^{(k)}$ *using the old pressure and density fields.*
- *Calculate the* $\mathbf{D}^{(k)}$ *fields.*
- *Solve the pressure correction equation.*
- *Correct* $\mathbf{u}^{(k)}$, P, *and* $\rho^{(k)}$.
- *Solve implicitly for* $r^{(k)}$.
- *Solve implicitly the energy equation and update the density fields.*
- *Solve the momentum equations explicitly and calculate the* $\mathbf{D}^{(k)}$ *fields.*
- *Solve the pressure correction equation.*
- *Correct* $\mathbf{u}^{(k)}$, P, *and* $\rho^{(k)}$.
- *Return to step one and iterate until convergence*

The MCBA following SIMPLEX (MCBA-SIMPLEX): Symbolic Form

Predictor:

$$\mathbf{u}_P^{(k)*} = \mathbf{HP}_P[\mathbf{u}^{(k)*}] - r^{(k)\circ}\mathbf{D}_P^{(k)} \nabla_P P^\circ \tag{140}$$

Corrector:

$$\left(\mathbf{u}^{(k)'}, P', \rho^{(k)'} \right) \left(\mathbf{u}^{(k)**} = \mathbf{u}^{(k)*} + \mathbf{u}^{(k)'}, P^* = P^\circ + P', \rho^{(k)*} = \rho^{(k)\circ} + \rho^{(k)'} \right. \tag{141}$$

$$\therefore \mathbf{u}_P^{(k)**} = \mathbf{HP}_P[\mathbf{u}^{(k)**}] - r^{(k)\circ}\mathbf{D}_P^{(k)} \nabla_P P^* = \mathbf{HP}_P[\mathbf{u}^{(k)*} + \mathbf{u}^{(k)'}] - r^{(k)\circ}\mathbf{D}_P^{(k)} \nabla_P \left(P^\circ + P' \right) \tag{142}$$

$$\therefore \begin{cases} \mathbf{u}_P^{(k)'} = \mathbf{HP}_P[\mathbf{u}^{(k)'}] - r^{(k)\circ}\mathbf{D}_P^{(k)} \nabla_P P' \\ \rho^{(k)'} = C_\rho^{(k)} P' \end{cases} \quad (143$$

Condition:

$$\sum_k \left\{ \frac{\left(r_P^{(k)\circ} \rho_P^{(k)*} \right) - \left(r_P^{(k)} \rho_P^{(k)} \right)^{Old}}{\delta t} \Omega + \Delta_P \left[r^{(k)\circ} \rho^{(k)*} \mathbf{u}^{(k)**} . \mathbf{S} \right] \right\} = 0 \tag{144}$$

$$\sum_k \left\{ \begin{array}{l} \dfrac{r_P^{(k)\circ}\left(\rho_P^{(k)\circ}+\rho_P^{(k)\prime}\right)-\left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega \\ +\Delta_P\left[r^{(k)\circ}\left(\rho^{(k)\circ}+\rho^{(k)\prime}\right)\left(\mathbf{u}^{(k)*}+\mathbf{u}^{(k)\prime}\right)\mathbf{S}\right] \end{array} \right\} = 0 \tag{145}$$

$$\therefore \sum_k \left\{ \begin{array}{l} \dfrac{r_P^{(k)\circ}\left(\rho_P^{(k)\circ}+\rho_P^{(k)\prime}\right)-\left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega \\ +\Delta_P\left[r^{(k)\circ}\rho^{(k)\prime}\mathbf{u}^{(k)*}.\mathbf{S}\right]+\Delta_P\left[r^{(k)\circ}\rho^{(k)\circ}\mathbf{u}^{(k)\prime}.\mathbf{S}\right] \end{array} \right\} = -\sum_k \left\{ \begin{array}{l} \Delta_P\left[r^{(k)\circ}\rho^{(k)\circ}\mathbf{u}^{(k)*}.\mathbf{S}\right] \\ +\Delta_P\left[r^{(k)\circ}\rho^{(k)\prime}\mathbf{u}^{(k)\prime}.\mathbf{S}\right] \end{array} \right\} \tag{146}$$

Approximation:

Neglect: $\Delta_P\left[r^{(k)\circ}\rho^{(k)\prime}\mathbf{u}^{(k)\prime}.\mathbf{S}\right.$ and let

$$\mathbf{u}_P^{(k)\prime} = \mathbf{HP}_P[\mathbf{u}^{(k)\prime}] - r_P^{(k)\circ}\mathbf{D}_P^{(k)}\nabla_P P' = -r_P^{(k)\circ}\mathbf{D}_P^{(k)SX}\nabla_P P' \tag{147}$$

$$\Rightarrow -r_P^{(k)\circ}\mathbf{D}_P^{(k)SX}\nabla_P P' = \mathbf{HP}_P[-r^{(k)\circ}\mathbf{D}^{(k)SX}\nabla_P P'] - r_P^{(k)\circ}\mathbf{D}_P^{(k)}\nabla_P P' \tag{148}$$

Assume that the pressure difference local to the velocity is representative of all pressure differences i.e. $\mathbf{HP}_P[-r^{(k)\circ}\mathbf{D}^{(k)SX}\nabla_P P'] = -\left(\nabla_P P'\right)\mathbf{HP}_P[r^{(k)\circ}\mathbf{D}^{(k)SX}]$, thus:

$$-r_P^{(k)\circ}\mathbf{D}_P^{(k)SX}\nabla_P P' = -\left(\nabla_P P'\right)\mathbf{HP}_P[r^{(k)\circ}\mathbf{D}^{(k)SX}] - r_P^{(k)\circ}\mathbf{D}_P^{(k)}\nabla_P P' \tag{149}$$

$$\Rightarrow r_P^{(k)\circ}\mathbf{D}_P^{(k)SX} = \mathbf{HP}_P[r^{(k)\circ}\mathbf{D}^{(k)SX}] + r_P^{(k)\circ}\mathbf{D}_P^{(k)} \tag{150}$$

Approximate Equation:

$$\Rightarrow \begin{array}{l} \sum_k \left\{ \dfrac{\Omega}{\delta t} r_P^{(k)\circ}C_\rho^{(k)}P_P' + \Delta_P\left[r^{(k)\circ}C_\rho^{(k)}U^{(k)*}P'\right] - \Delta_P\left[r^{(k)\circ}\rho^{(k)\circ}\left(r^{(k)\circ}\mathbf{D}^{(k)SX}\nabla P'\right)\mathbf{S}\right] \right\} \\ = -\sum_k \left\{ \dfrac{r_P^{(k)\circ}\rho_P^{(k)\circ} - \left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega + \Delta_P\left[r^{(k)\circ}\rho^{(k)\circ}U^{(k)*}\right] \right\} \end{array} \tag{151}$$

A Global MCBA-SIMPLEX Iteration

- *Solve implicitly for* $\mathbf{u}^{(k)}$*, using the old pressure and density fields.*
- *Calculate the* $\mathbf{D}^{(k)}$ *fields.*
- *Solve implicitly for the* $\mathbf{D}^{(k)SX}$ *fields.*
- *Solve the pressure correction equation using these* $\mathbf{D}^{(k)SX}$ *fields.*
- *Correct* $\mathbf{u}^{(k)}$*, P and* $\rho^{(k)}$*.*
- *Solve implicitly for* $r^{(k)}$*.*
- *Solve implicitly the energy equations and update the density fields.*
- *Return to the first step and iterate until convergence.*

The MCBA following SIMPLEM (MCBA-SIMPLEM): Symbolic Form

First Predictor:

No predictor stage. Only coefficients of the momentum equations are calculated.

First Corrector:

$$\left(\mathbf{u}^{(k)'}, P', \rho^{(k)'}\right)\left(\mathbf{u}^{(k)*} = \mathbf{u}^{(k)\circ} + \mathbf{u}^{(k)'}, P^* = P^\circ + P', \rho^{(k)*} = \rho^{(k)\circ} + \rho^{(k)'}\right. \tag{152}$$

$$\therefore \mathbf{u}_P^{(k)*} = \mathbf{HP}_P[\mathbf{u}^{(k)*}] - r^{(k)\circ}\mathbf{D}_P^{(k)}\nabla_P P^* \tag{153}$$

$$\mathbf{u}_P^{(k)\circ} = \mathbf{HP}_P[\mathbf{u}^{(k)\circ}] - r^{(k)\circ}\mathbf{D}_P^{(k)}\nabla_P P^\circ \tag{154}$$

$$\therefore \begin{cases} \mathbf{u}_P^{(k)'} = \mathbf{HP}_P[\mathbf{u}^{(k)'}] - r^{(k)\circ}\mathbf{D}_P^{(k)}\nabla_P P' \\ \rho^{(k)'} = C_\rho^{(k)} P' \end{cases} \tag{155}$$

Condition:

$$\sum_k \left\{ \frac{\left(r_P^{(k)\circ}\rho_P^{(k)*}\right) - \left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega + \Delta_P\left[r^{(k)\circ}\rho^{(k)*}\mathbf{u}^{(k)*}.\mathbf{S}\right] \right\} = 0 \tag{156}$$

$$\therefore \sum_k \left\{ \begin{array}{l} \dfrac{r_P^{(k)\circ}\left(\rho_P^{(k)\circ} + \rho_P^{(k)'}\right) - \left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega \\ \quad + \Delta_P\left[r^{(k)\circ}\left(\rho^{(k)\circ} + \rho^{(k)'}\right)\left(\mathbf{u}^{(k)\circ} + \mathbf{HP}\left[\mathbf{u}^{(k)'}\right] - r^{(k)\circ}\mathbf{D}^{(k)}\nabla P'\right)\mathbf{S}\right] \end{array} \right\} = 0 \tag{157}$$

$$\therefore \begin{array}{l} \displaystyle\sum_k \left\{ \frac{\Omega}{\delta t} r_P^{(k)\circ} C_\rho^{(k)} P'_P + \Delta_P\left[r^{(k)\circ} C_\rho^{(k)} U^{(k)\circ} P'\right] - \Delta_P\left[r^{(k)\circ}\rho^{(k)\circ}\left(r^{(k)\circ}\mathbf{D}^{(k)}\nabla P'\right)\mathbf{S}\right] \right\} \\ \displaystyle\qquad = -\sum_k \left\{ \begin{array}{l} \dfrac{r_P^{(k)\circ}\rho_P^{(k)\circ} - \left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega + \Delta_P\left[r^{(k)\circ}\rho^{(k)\circ}U^{(k)\circ}\right] \\ + \Delta_P\left[r^{(k)\circ}\rho^{(k)\circ}\left(\mathbf{HP}\left[\mathbf{u}^{(k)'}\right]\right)\mathbf{S}\right] + \Delta_P\left[r^{(k)\circ}\rho^{(k)'}\mathbf{u}^{(k)'}.\mathbf{S}\right] \end{array} \right\} \end{array} \tag{158}$$

Approximation:

Neglect: $\mathbf{HP}\left[\mathbf{u}^{(k)'}\right.$, $\Delta_P\left[r^{(k)\circ}\rho^{(k)'}\mathbf{u}^{(k)'}.\mathbf{S}\right.$

$$\Rightarrow \mathbf{u}_P^{(k)'} = -r^{(k)\circ}\mathbf{D}_P^{(k)}\nabla_P P' \tag{159}$$

Approximate Equation:

$$\sum_k \left\{ \frac{\Omega}{\delta t} r_P^{(k)\circ} C_\rho^{(k)} P_P' + \Delta_P \left[r^{(k)\circ} C_\rho^{(k)} U^{(k)\circ} P' \right] - \Delta_P \left[r^{(k)\circ} \rho^{(k)\circ} \left(r^{(k)\circ} \mathbf{D}^{(k)} \nabla P' \right) \mathbf{S} \right] \right\}$$

$$\Rightarrow \qquad = -\sum_k \left\{ \frac{r_P^{(k)\circ} \rho_P^{(k)\circ} - \left(r_P^{(k)} \rho_P^{(k)} \right)^{Old}}{\delta t} \Omega + \Delta_P \left[r^{(k)\circ} \rho^{(k)\circ} U^{(k)\circ} \right] \right\}$$

Second Predictor:

$$\mathbf{u}_P^{(k)**} = \mathbf{HP}_P[\mathbf{u}^{(k)**}] - r^{(k)\circ} \mathbf{D}_P^{(k)*} \nabla_P P^* \tag{160}$$

Second Corrector:

No corrector stage.

A Global MCBA-SIMPLEM Iteration

- *Calculate the* $\mathbf{D}^{(k)}$ *fields based on values from the previous iteration.*
- *Solve the pressure correction equation.*
- *Correct* $\mathbf{u}^{(k)}$, P *and* $\rho^{(k)}$.
- *Calculate new* $\mathbf{HP}^{(k)}$ *and* $\mathbf{D}^{(k)}$ *fields.*
- *Solve implicitly for* $\mathbf{u}^{(k)}$ *using the new fields.*
- *Solve implicitly for* $r^{(k)}$.
- *Solve implicitly the energy equations and update the density fields.*
- *Return to the first step and iterate until convergence.*

Discussion of the MCBA following the SIMPLE-like segregated Approach

From the above derivations it is clear that the main difference among the various MCBA is in the approximation made to the $\mathbf{HP}[\mathbf{u}^{(k)'}]$ term. Since at the state of convergence the pressure and velocity correction fields are zero, the approximation introduced does not have any effect on the final solution. Rather, it only affects the convergence behavior.

As mentioned before, the majority of SIMPLE-like algorithms have not been extended and tested in multi-fluid flow situations. As such, the only way to discuss their merit is by assuming that they behave the same way as in single-fluid flow situations.

In the MCBA-SIMPLE algorithm, the $\mathbf{HP}[\mathbf{u}^{(k)'}]$ terms are dropped. Because of this, the predicted pressure correction field is overestimated and the corrected velocity field does not satisfy the momentum equations. Therefore, in order to avoid divergence, the pressure field is

under-relaxed. This under-relaxation procedure has resulted in the rate of convergence to be greatly dependent on the proper choice of the under-relaxation factors for the velocity components and the pressure.

The SIMPLEC algorithm of Van Doormaal and Raithby [18] was developed with the intention of alleviating the aforementioned problem through a better approximation to the $\mathbf{HP}[\mathbf{u}^{(k)'}]$ (i.e. neglecting $\mathbf{HP}[\mathbf{u}^{(k)'} - \mathbf{u}_{\mathbf{P}}^{(k)'}]$ rather than $\mathbf{HP}[\mathbf{u}^{(k)'}]$), so as to eliminate the need for under-relaxing the pressure field. Consequently, a higher rate of convergence is expected in the MCBA-SIMPLEC.

In the MCBA-PRIME algorithm [72], the momentum equations are solved explicitly. This explicit treatment is justified by the small contribution to the convergence of the entire flow field by the iterative sweeps within each momentum equation. On the other hand, finding the correct solution for the pressure field represents the most important factor in the overall convergence. Moreover, the terms neglected in MCBA-PRIME $\left(\mathbf{HP}_P\left[\mathbf{u}^{(k)'}\right] + \mathbf{HP}_P\left[\mathbf{u}^{(k)*} - \mathbf{u}^{(k)\circ}\right]\right.$ can become smaller than the term neglected in MCBA-SIMPLE $\left(\mathbf{HP}_P\left[\mathbf{u}^{(k)'}\right]\right.$ if $\mathbf{HP}_P\left[\mathbf{u}^{(k)'}\right.$ and $\mathbf{HP}_P\left[\mathbf{u}^{(k)*} - \mathbf{u}^{(k)\circ}\right.$ are of opposite signs. It is worth noting that $\mathbf{HP}_P\left[\mathbf{u}^{(k)'}\right. = \mathbf{HP}_P\left[\mathbf{u}^{(k)**} - \mathbf{u}^{(k)*}\right.$ is a correction to satisfy continuity, while $\mathbf{HP}_P\left[\mathbf{u}^{(k)*} - \mathbf{u}^{(k)\circ}\right.$ is a correction to satisfy momentum. Usually the correction added to satisfy momentum is opposite to that added to satisfy continuity and hence, the neglected term $\left(\mathbf{HP}_P\left[\mathbf{u}^{(k)'}\right] + \mathbf{HP}_P\left[\mathbf{u}^{(k)*} - \mathbf{u}^{(k)\circ}\right]\right.$ is smaller.

In MCBA-SIMPLEST, the coefficients of the momentum equations are separated into their diffusion and convection parts as:

$$\mathbf{HP}_P\left[\mathbf{u}^{(k)}\right] = \mathbf{HP}_P^D\left[\mathbf{u}^{(k)}\right] + \mathbf{HP}_P^C\left[\mathbf{u}^{(k)}\right. \tag{161}$$

The momentum equations are then solved treating the diffusion terms implicitly and the convection terms explicitly. This procedure is justified by the fact that disturbances in a pure

diffusion situation propagate instantaneously in all directions, but their amplitude decays rapidly. This is equivalent to propagation of errors throughout the entire solution domain, in a single iteration by implicit solution methods. On the other hand, disturbances in a pure convection situation are propagated, in the flow direction, at a finite speed without any change in their magnitude. This is similar to propagation of error, from a particular point to the neighboring grid points, in a single iteration of explicit iterative methods. This justifies the implicit and explicit treatments of the diffusion and convection parts, respectively. Moreover, similar to MCBA-PRIME, the terms neglected in MCBA-SIMPLEST can become smaller than the term neglected in MCBA-SIMPLE. Therefore, the MCBA-SIMPLEST can be seen to be a compromise between MCBA-SIMPLE and MCBA-PRIME.

As presented, the MCBA-SIMPLER can be viewed as a combination of an approximate form of MCBA-PRIME to compute the pressure and MCBA-SIMPLE to compute the velocities, thereby combining the best features of both methods.

The MCBA-PISO consists of a predictor step and one or more corrector steps. The use of the additional corrector steps bring the velocity and pressure fields closer to satisfying both the momentum and continuity equations. Moreover, by following the sequence of events, it can be easily seen that MCBA-PISO may be considered to be a combination of one MCBA-SIMPLE step and one MCBA-PRIME step, hence combining the implicitness of the SIMPLE algorithm with the stability of the PRIME algorithm.

In all MCBA-SIMPLE-based methods, no care is taken to ensure that the rate of convergence will not degrade with grid refinement. This concern is addressed in MCBA-SIMPLEX by using extrapolation to express all pressure differences in the domain in terms of the pressure difference local to the velocity. The idea is based on the fact that the spatial distribution of pressure difference influence changes little with grid refinement. Therefore, if the pressure difference influence were restricted to a control volume, it would be appropriate to assume

that, by extrapolation, the pressure difference at the main grid point adequately represents the pressure differences at the control volume faces. Moreover, the pressure correction equation in MCBA-SIMPLEX involves the $\mathbf{D}^{(k)sx}$ fields, rather than the $\mathbf{D}^{(k)}$ fields, which have to be computed by solving an additional system of equations (Eq. (150)).

The pressure field in MCBA-SIMPLEM is computed using the old velocity fields. This is nearly equivalent to MCBA-PRIME, which does a good job in correcting the pressure field. By so doing however, the velocity corrections will be at a disadvantage. Therefore, in MCBA-SIMPLEM the disadvantages and advantages of MCBA-SIMPLE are interchanged.

The Expanded Form of the Pressure-Correction Equation

It is obvious by now that the various simplified pressure-correction equations are similar and may be written as:

$$\sum_k \left\{ \frac{\Omega}{\delta t} r_P^{(k)\circ} C_\rho^{(k)} P_P' + \Delta_P\left[r^{(k)\circ} C_\rho^{(k)} U^{(k)} P' \right] - \Delta_P\left[r^{(k)^\circ} \rho^{(k)} \left(r^{(k)^\circ} \mathbf{D}^{(k)} \nabla P' \right) \mathbf{S} \right] \right\}$$
$$= -\sum_k \left\{ \frac{ r_P^{(k)\circ} \rho_P^{(k)} - \left(r_P^{(k)} \rho_P^{(k)} \right)^{Old} }{\delta t} \Omega + \Delta_P\left[r^{(k)\circ} \rho^{(k)} U^{(k)} \right] \right\} \tag{162}$$

Where, depending on the algorithm used, $U^{(k)}$ and $\rho^{(k)}$ represent values from the previous iteration or from a previous corrector step. When discretizing this equation, careful attention should be paid to the second term on the left hand side that is similar to a convection term and for which any convective scheme may be used. Adopting the UPWIND scheme [16] the discretized form of the convection-like term $\Delta_P\left[r_P^{(k)^\circ} C_\rho^{(k)} U^{(k)} P' \right.$ is:

$$\begin{aligned}
\Delta_P\left[r_P^{(k)^\circ} C_\rho^{(k)} U^{(k)} P'\right] &= \left(r_P^{(k)^\circ} C_\rho^{(k)} U^{(k)} P'\right)_e + \left(r_P^{(k)^\circ} C_\rho^{(k)} U^{(k)} P\right)_w + \left(r_P^{(k)^\circ} C_\rho^{(k)} U^{(k)} P\right)_n \\
&\quad + \left(r_P^{(k)^\circ} C_\rho^{(k)} U^{(k)} P\right)_s + \left(r_P^{(k)^\circ} C_\rho^{(k)} U^{(k)} P\right)_t + \left(r_P^{(k)^\circ} C_\rho^{(k)} U^{(k)} P'\right)_b \\
&= \left(r^{(k)^\circ} C_\rho^{(k)}\right)_e \left[\left\| U_e^{(k)}, 0\right\| P_P' - \left\| -U_e^{(k)}, 0\right\| P_E'\right] + \left(r^{(k)^\circ} C_\rho^{(k)}\right)_w \left[\left\| U_w^{(k)}, 0\right\| P_P' - \left\| -U_w^{(k)}, 0\right\| P_W'\right] \\
&\quad + \left(r^{(k)^\circ} C_\rho^{(k)}\right)_n \left[\left\| U_n^{(k)}, 0\right\| P_P' - \left\| -U_n^{(k)}, 0\right\| P_N'\right] + \left(r^{(k)^\circ} C_\rho^{(k)}\right)_s \left[\left\| U_s^{(k)}, 0\right\| P_P' - \left\| -U_s^{(k)}, 0\right\| P_S'\right] \\
&\quad + \left(r^{(k)^\circ} C_\rho^{(k)}\right)_t \left[\left\| U_t^{(k)}, 0\right\| P_P' - \left\| -U_t^{(k)}, 0\right\| P_T'\right] + \left(r^{(k)^\circ} C_\rho^{(k)}\right)_b \left[\left\| U_b^{(k)}, 0\right\| P_P' - \left\| -U_b^{(k)}, 0\right\| P_B'\right]
\end{aligned} \tag{163}$$

Rearranging, one obtains:

$$\begin{aligned}
\Delta_P\left[r_P^{(k)^\circ} C_\rho^{(k)} U^{(k)} P'\right] &= \begin{bmatrix} \left(r^{(k)^\circ} C_\rho^{(k)}\right)_e \left\| U_e^{(k)}, 0\right\| + \left(r^{(k)^\circ} C_\rho^{(k)}\right)_w \left\| U_w^{(k)}, 0\right\| \\ + \left(r^{(k)^\circ} C_\rho^{(k)}\right)_n \left\| U_n^{(k)}, 0\right\| + \left(r^{(k)^\circ} C_\rho^{(k)}\right)_s \left\| U_s^{(k)}, 0\right\| \\ + \left(r^{(k)^\circ} C_\rho^{(k)}\right)_t \left\| U_t^{(k)}, 0\right\| + \left(r^{(k)^\circ} C_\rho^{(k)}\right)_b \left\| U_b^{(k)}, 0\right\| \end{bmatrix} P_P' \\
&\quad - \left(r^{(k)^\circ} C_\rho^{(k)}\right)_e \left\| -U_e^{(k)}, 0\right\| P_E' - \left(r^{(k)^\circ} C_\rho^{(k)}\right)_w \left\| -U_w^{(k)}, 0\right\| P_W' \\
&\quad - \left(r^{(k)^\circ} C_\rho^{(k)}\right)_n \left\| -U_n^{(k)}, 0\right\| P_N' - \left(r^{(k)^\circ} C_\rho^{(k)}\right)_s \left\| -U_s^{(k)}, 0\right\| P_S' \\
&\quad - \left(r^{(k)^\circ} C_\rho^{(k)}\right)_t \left\| -U_t^{(k)}, 0\right\| P_T' - \left(r^{(k)^\circ} C_\rho^{(k)}\right)_b \left\| -U_b^{(k)}, 0\right\| P_B'
\end{aligned} \tag{164}$$

The term $\Delta_P\left[r^{(k)^\circ} \rho^{(k)*}\left(-D_P^{(k)} \nabla P'\right).\mathbf{S}\right]$ is discretized following the same procedure that was used in discretizing the diffusion flux. Its final form is given by:

$$\begin{aligned}
\Delta_P\left[r^{(k)^\circ} \rho^{(k)}\left(r^{(k)^\circ} \mathbf{D}^{(k)} \nabla P'\right).\mathbf{S}\right] &= \left(r^{(k)^\circ} \rho^{(k)}\left(r^{(k)^\circ} \mathbf{D}^{(k)} \nabla P'\right).\mathbf{S}\right)_e + \left(r^{(k)^\circ} \rho^{(k)}\left(r^{(k)^\circ} \mathbf{D}^{(k)} \nabla P'\right).\mathbf{S}\right)_w \\
&\quad + \left(r^{(k)^\circ} \rho^{(k)}\left(r^{(k)^\circ} \mathbf{D}^{(k)} \nabla P'\right).\mathbf{S}\right)_n + \left(r^{(k)^\circ} \rho^{(k)}\left(r^{(k)^\circ} \mathbf{D}^{(k)} \nabla P'\right).\mathbf{S}\right)_s \\
&\quad + \left(r^{(k)^\circ} \rho^{(k)}\left(r^{(k)^\circ} \mathbf{D}^{(k)} \nabla P'\right).\mathbf{S}\right)_t + \left(r^{(k)^\circ} \rho^{(k)}\left(r^{(k)^\circ} \mathbf{D}^{(k)} \nabla P'\right).\mathbf{S}\right)_b
\end{aligned} \tag{165}$$

Or in expanded form as,

$$
\begin{aligned}
&\Delta_P\left[r^{(k)\circ}\rho^{(k)}\left(r^{(k)\circ}\mathbf{D}^{(k)}\nabla P'\right)\mathbf{S}\right] \\
&= r_e^{(k)\circ} r_e^{(k)\circ} \rho_e^{(k)} \frac{\overline{\mathrm{D}}_e^{(k)}[u]\left(S_e^x\right)^2 + \overline{\mathrm{D}}_e^{(k)}[v]\left(S_e^y\right)^2 + \overline{\mathrm{D}}_e^{(k)}[w]\left(S_e^z\right)^2}{\mathbf{S}_e.\mathbf{d}_e}\left(P_E' - P_P'\right) \\
&\underline{+r_e^{(k)\circ} r_e^{(k)\circ} \rho_e^{(k)}\left(\overline{\mathrm{D}}_e^{(k)}[u]\left(\overline{\nabla P'}\right)_e^x \kappa_e^x + \overline{\mathrm{D}}_e^{(k)}[v]\left(\overline{\nabla P'}\right)_e^y \kappa_e^y + \overline{\mathrm{D}}_e^{(k)}[w]\left(\overline{\nabla P'}\right)_e^z \kappa_e^z\right)} \\
&+r_w^{(k)\circ} r_w^{(k)\circ} \rho_w^{(k)} \frac{\overline{\mathrm{D}}_w^{(k)}[u]\left(S_w^x\right)^2 + \overline{\mathrm{D}}_w^{(k)}[v]\left(S_w^y\right)^2 + \overline{\mathrm{D}}_w^{(k)}[w]\left(S_w^z\right)^2}{\mathbf{S}_w.\mathbf{d}_w}\left(P_W' - P_P'\right) \\
&\underline{+r_w^{(k)\circ} r_w^{(k)\circ} \rho_w^{(k)}\left(\overline{\mathrm{D}}_w^{(k)}[u]\left(\overline{\nabla P'}\right)_w^x \kappa_w^x + \overline{\mathrm{D}}_w^{(k)}[v]\left(\overline{\nabla P'}\right)_w^y \kappa_w^y + \overline{\mathrm{D}}_w^{(k)}[w]\left(\overline{\nabla P'}\right)_w^z \kappa_w^z\right.} \\
&+r_n^{(k)\circ} r_n^{(k)\circ} \rho_n^{(k)} \frac{\overline{\mathrm{D}}_n^{(k)}[u]\left(S_n^x\right)^2 + \overline{\mathrm{D}}_n^{(k)}[v]\left(S_n^y\right)^2 + \overline{\mathrm{D}}_n^{(k)}[w]\left(S_n^z\right)^2}{\mathbf{S}_n.\mathbf{d}_n}\left(P_N' - P_P'\right) \\
&\underline{+r_n^{(k)\circ} r_n^{(k)\circ} \rho_n^{(k)}\left(\overline{\mathrm{D}}_n^{(k)}[u]\left(\overline{\nabla P'}\right)_n^x \kappa_n^x + \overline{\mathrm{D}}_n^{(k)}[v]\left(\overline{\nabla P'}\right)_n^y \kappa_n^y + \overline{\mathrm{D}}_n^{(k)}[w]\left(\overline{\nabla P'}\right)_n^z \kappa_n^z\right)} \\
&+r_s^{(k)\circ} r_s^{(k)\circ} \rho_s^{(k)} \frac{\overline{\mathrm{D}}_s^{(k)}[u]\left(S_s^x\right)^2 + \overline{\mathrm{D}}_s^{(k)}[v]\left(S_s^y\right)^2 + \overline{\mathrm{D}}_s^{(k)}[w]\left(S_s^z\right)^2}{\mathbf{S}_s.\mathbf{d}_s}\left(P_S' - P_P'\right) \\
&\underline{+r_s^{(k)\circ} r_s^{(k)\circ} \rho_s^{(k)}\left(\overline{\mathrm{D}}_s^{(k)}[u]\left(\overline{\nabla P'}\right)_s^x \kappa_s^x + \overline{\mathrm{D}}_s^{(k)}[v]\left(\overline{\nabla P'}\right)_s^y \kappa_s^y + \overline{\mathrm{D}}_s^{(k)}[w]\left(\overline{\nabla P'}\right)_s^z \kappa_s^z\right)} \\
&+r_t^{(k)\circ} r_t^{(k)\circ} \rho_t^{(k)} \frac{\overline{\mathrm{D}}_t^{(k)}[u]\left(S_t^x\right)^2 + \overline{\mathrm{D}}_t^{(k)}[v]\left(S_t^y\right)^2 + \overline{\mathrm{D}}_t^{(k)}[w]\left(S_t^z\right)^2}{\mathbf{S}_t.\mathbf{d}_t}\left(P_T' - P_P'\right) \\
&\underline{+r_t^{(k)\circ} r_t^{(k)\circ} \rho_t^{(k)}\left(\overline{\mathrm{D}}_t^{(k)}[u]\left(\overline{\nabla P'}\right)_t^x \kappa_t^x + \overline{\mathrm{D}}_t^{(k)}[v]\left(\overline{\nabla P'}\right)_t^y \kappa_t^y + \overline{\mathrm{D}}_t^{(k)}[w]\left(\overline{\nabla P'}\right)_t^z \kappa_t^z\right)} \\
&+r_b^{(k)\circ} r_b^{(k)\circ} \rho_b^{(k)} \frac{\overline{\mathrm{D}}_b^{(k)}[u]\left(S_b^x\right)^2 + \overline{\mathrm{D}}_b^{(k)}[v]\left(S_b^y\right)^2 + \overline{\mathrm{D}}_b^{(k)}[w]\left(S_b^z\right)^2}{\mathbf{S}_b.\mathbf{d}_b}\left(P_B' - P_P'\right) \\
&\underline{+r_b^{(k)\circ} r_b^{(k)\circ} \rho_b^{(k)}\left(\overline{\mathrm{D}}_b^{(k)}[u]\left(\overline{\nabla P'}\right)_b^x \kappa_b^x + \overline{\mathrm{D}}_b^{(k)}[v]\left(\overline{\nabla P'}\right)_b^y \kappa_b^y + \overline{\mathrm{D}}_b^{(k)}[w]\left(\overline{\nabla P'}\right)_b^z \kappa_b^z\right)}
\end{aligned}
\tag{166}
$$

where the underlined terms account for the non-orthogonal factors. They are usually neglected since their contribution is small in comparison with other terms and vanish when the grid is orthogonal. However, they could be accounted for by moving them to the right hand side, adding them to the source term, and modifying the solver to explicitly update their values after a solver (not global) iteration. Neglecting these terms, Eq. (166) becomes:

$$
\begin{aligned}
&\Delta_P\left[r^{(k)^\circ}\rho^{(k)}\left(r^{(k)^\circ}\mathbf{D}^{(k)}\nabla P'\right)\mathbf{S}\right] \\
&= r_e^{(k)^\circ} r_e^{(k)^\circ} \rho_e^{(k)} \frac{\overline{\mathrm{D}}_e^{(k)}[u]\left(S_e^x\right)^2 + \overline{\mathrm{D}}_e^{(k)}[v]\left(S_e^y\right)^2 + \overline{\mathrm{D}}_e^{(k)}[w]\left(S_e^z\right)^2}{\mathbf{S}_e.\mathbf{d}_e}\left(P_E' - P_P'\right) \\
&+ r_w^{(k)^\circ} r_w^{(k)^\circ} \rho_w^{(k)} \frac{\overline{\mathrm{D}}_w^{(k)}[u]\left(S_w^x\right)^2 + \overline{\mathrm{D}}_w^{(k)}[v]\left(S_w^y\right)^2 + \overline{\mathrm{D}}_w^{(k)}[w]\left(S_w^z\right)^2}{\mathbf{S}_w.\mathbf{d}_w}\left(P_W' - P_P'\right. \\
&+ r_n^{(k)^\circ} r_n^{(k)^\circ} \rho_n^{(k)} \frac{\overline{\mathrm{D}}_n^{(k)}[u]\left(S_n^x\right)^2 + \overline{\mathrm{D}}_n^{(k)}[v]\left(S_n^y\right)^2 + \overline{\mathrm{D}}_n^{(k)}[w]\left(S_n^z\right)^2}{\mathbf{S}_n.\mathbf{d}_n}\left(P_N' - P_P'\right) \\
&+ r_s^{(k)^\circ} r_s^{(k)^\circ} \rho_s^{(k)} \frac{\overline{\mathrm{D}}_s^{(k)}[u]\left(S_s^x\right)^2 + \overline{\mathrm{D}}_s^{(k)}[v]\left(S_s^y\right)^2 + \overline{\mathrm{D}}_s^{(k)}[w]\left(S_s^z\right)^2}{\mathbf{S}_s.\mathbf{d}_s}\left(P_S' - P_P'\right) \\
&+ r_t^{(k)^\circ} r_t^{(k)^\circ} \rho_t^{(k)} \frac{\overline{\mathrm{D}}_t^{(k)}[u]\left(S_t^x\right)^2 + \overline{\mathrm{D}}_t^{(k)}[v]\left(S_t^y\right)^2 + \overline{\mathrm{D}}_t^{(k)}[w]\left(S_t^z\right)^2}{\mathbf{S}_t.\mathbf{d}_t}\left(P_T' - P_P'\right) \\
&+ r_b^{(k)^\circ} r_b^{(k)^\circ} \rho_b^{(k)} \frac{\overline{\mathrm{D}}_b^{(k)}[u]\left(S_b^x\right)^2 + \overline{\mathrm{D}}_b^{(k)}[v]\left(S_b^y\right)^2 + \overline{\mathrm{D}}_b^{(k)}[w]\left(S_b^z\right)^2}{\mathbf{S}_b.\mathbf{d}_b}\left(P_B' - P_P'\right) \\
&= \Gamma_e^{(k)}\left(P_E' - P_P'\right) + \Gamma_w^{(k)}\left(P_W' - P_P'\right) + \Gamma_n^{(k)}\left(P_N' - P_P'\right) + \Gamma_s^{(k)}\left(P_S' - P_P'\right) \\
&+ \Gamma_T^{(k)}\left(P_T' - P_P'\right) + \Gamma_b^{(k)}\left(P_B' - P_P'\right) \\
&= \left(\Gamma_e^{(k)} + \Gamma_w^{(k)} + \Gamma_n^{(k)} + \Gamma_s^{(k)} + \Gamma_t^{(k)} + \Gamma_b^{(k)}\right)P_P' \\
&- \Gamma_e^{(k)}P_E' - \Gamma_w^{(k)}P_W' - \Gamma_n^{(k)}P_N' - \Gamma_s^{(k)}P_S' - \Gamma_t^{(k)}P_T' - \Gamma_b^{(k)}P_B'
\end{aligned}
\tag{167}
$$

Moreover, the discretized form of $\Delta_P\left[r^{(k)^\circ}\rho^{(k)}U^{(k)}\right.$ is given by:

$$
\begin{aligned}
\Delta_P\left[r^{(k)^\circ}\rho^{(k)}U^{(k)}\right] = r_e^{(k)^\circ}\rho_e^{(k)}U_e^{(k)} + r_w^{(k)^\circ}\rho_w^{(k)}U_w^{(k)} + r_n^{(k)^\circ}\rho_n^{(k)}U_n^{(k)} + r_s^{(k)^\circ}\rho_s^{(k)}U_s^{(k)} \\
+ r_t^{(k)^\circ}\rho_t^{(k)}U_t^{(k)} + r_b^{(k)^\circ}\rho_b^{(k)}U_b^{(k)}
\end{aligned}
\tag{168}
$$

Substituting the various terms in Eq. (162) by their equivalent expressions as derived above, the final form of the pressure-correction equation is written as:

$$
A_P^{P'}P_P' = A_E^{P'}P_E' + A_W^{P'}P_W' + A_N^{P'}P_N' + A_S^{P'}P_S' + A_T^{P'}P_T' + A_B^{P'}P_B' + B_P^{P'}
\tag{169}
$$

where

$$
\begin{aligned}
A_E^{P'} &= \sum_k \left(\Gamma_e^{(k)} + \left(r^{(k)\circ} C_\rho^{(k)}\right)_e \left\|-U_e^{(k)}, 0\right\|\right) \\
A_W^{P'} &= \sum_k \left(\Gamma_w^{(k)} + \left(r^{(k)\circ} C_\rho^{(k)}\right)_w \left\|-U_w^{(k)}, 0\right\|\right) \\
A_N^{P'} &= \sum_k \left(\Gamma_n^{(k)} + \left(r^{(k)\circ} C_\rho^{(k)}\right)_n \left\|-U_n^{(k)}, 0\right\|\right) \\
A_S^{P'} &= \sum_k \left(\Gamma_s^{(k)} + \left(r^{(k)\circ} C_\rho^{(k)}\right)_s \left\|-U_s^{(k)}, 0\right\|\right) \\
A_T^{P'} &= \sum_k \left(\Gamma_t^{(k)} + \left(r^{(k)\circ} C_\rho^{(k)}\right)_t \left\|-U_t^{(k)}, 0\right\|\right) \\
A_B^{P'} &= \sum_k \left(\Gamma_b^{(k)} + \left(r^{(k)\circ} C_\rho^{(k)}\right)_b \left\|-U_b^{(k)}, 0\right\|\right) \\
A_P^{P'} &= A_E^{P'} + A_W^{P'} + A_N^{P'} + A_S^{P'} + A_T^{P'} + A_B^{P'} + \\
&\sum_k \left(\begin{array}{l} \dfrac{\left(r^{(k)\circ} C_\rho^{(k)}\right)_P \Omega_P}{\delta t} + \left(r^{(k)\circ} C_\rho^{(k)}\right)_e U_e^{(k)} + \left(r^{(k)\circ} C_\rho^{(k)}\right)_w U_w^{(k)} + \left(r^{(k)\circ} C_\rho^{(k)}\right)_n U_n^{(k)} \\ \qquad + \left(r^{(k)\circ} C_\rho^{(k)}\right)_s U_s^{(k)} + \left(r^{(k)\circ} C_\rho^{(k)}\right)_t U_t^{(k)} + \left(r^{(k)\circ} C_\rho^{(k)}\right)_b U_b^{(k)} \end{array} \right) \\
B_P^{P'} &= -\sum_k \left\{ \frac{\left(r_P^{(k)\circ} \rho_P^{(k)} - r_P^{(k)old} \rho_P^{(k)old}\right)}{\delta t} \Omega_P + \left(\begin{array}{l} r_e^{(k)} \rho_e^{(k)} U_e^{(k)} + r_w^{(k)} \rho_w^{(k)} U_w^{(k)} + r_n^{(k)} \rho_n^{(k)} U_n^{(k)} \\ + r_s^{(k)} \rho_s^{(k)} U_s^{(k)} + r_t^{(k)} \rho_t^{(k)} U_t^{(k)} + r_b^{(k)} \rho_b^{(k)} U_b^{(k)} \end{array} \right) \right\}
\end{aligned}
\tag{170}
$$

The corrections are then applied to the velocity, pressure, and density fields using the following equations:

$$
\begin{aligned}
\mathbf{u}_P^{(k)*} &= \mathbf{u}_P^{(k)\circ} - r^{(k)\circ} \mathbf{D}_P^{(k)} \nabla_P P' \\
P^* &= P^\circ + P' \\
\rho^{(k)*} &= \rho^{(k)\circ} + \rho^{(k)'} = \rho^{(k)\circ} + C_\rho^{(k)} P'
\end{aligned}
\tag{171}
$$

Improvement #2: Weighted Pressure Correction

Numerical experiments [73] using the above approach to simulate air-water flows have shown poor conservation of the lighter fluid. To understand this behavior, the residual error of the kth phase continuity equation, after any global iteration, that arises because the velocity, density, and volume fraction fields do not yet satisfy the continuity equation is denoted by RESC$^{(k)}$. The pressure correction equation being derived from the global conservation equation, it is intended to correct the velocity fields so as to drive the global

residual error, which is equal to the sum of the local residuals, to zero i.e. $RESC^{(1)} + RESC^{(2)} + .. + RESC^{(n)} \rightarrow 0$.

In the presence of a relatively very high density fluid such that $\rho^{(n)} >> \rho^{(k)}$ for k n, the residual error of the nth fluid will be of a magnitude commensurate with the respective phase density, i.e. $RESC^{(n)}$ is expected to be much larger than $RESC^{(k)}$ for k n. In this case, only the residual of the high density fluid will be significant while that of the low density fluid will be relatively negligible, and hence the pressure correction will tend to drive the high density fluid to conservation.

This problem can be considerably alleviated by normalizing the individual continuity equations, and hence the global mass conservation equation, by means of a weighting factor such as a reference density $\underline{\rho}^{(k)}$ (which is fluid dependent) to give a volumetric conservation equation of the form:

$$\sum_k \left\{ \frac{\left(r_P^{(k)} \rho_P^{(k)} / \underline{\rho}^{(k)} \right) - \left(r_P^{(k)} \rho_P^{(k)} / \underline{\rho}^{(k)} \right)^{Old}}{\delta t} \Omega_P + \Delta_P \left[r^{(k)} \left(\frac{\rho^{(k)}}{\underline{\rho}^{(k)}} \right) \mathbf{u}^{(k)} . \mathbf{S} \right] \right\} = 0 \tag{172}$$

In this case the pressure correction equation becomes

$$\begin{aligned} \sum_k \left\{ \frac{\Omega}{\delta t \underline{\rho}^{(k)}} r_P^{(k)\circ} C_\rho^{(k)} P_P' + \Delta_P \left[\frac{r^{(k)\circ} C_\rho^{(k)} U^{(k)} P'}{\underline{\rho}^{(k)}} \right] - \Delta_P \left[r^{(k)^\circ} \frac{\rho^{(k)}}{\underline{\rho}^{(k)}} \left(r^{(k)^\circ} \mathbf{D}^{(k)} \nabla P' \right) \mathbf{S} \right] \right\} \\ = -\sum_k \left\{ \begin{aligned} & \frac{r_P^{(k)\circ} \left(\rho_P^{(k)} / \underline{\rho}^{(k)} \right) - \left[r_P^{(k)} \left(\rho_P^{(k)} / \underline{\rho}^{(k)} \right) \right]^{Old}}{\delta t} \Omega + \Delta_P \left[r^{(k)\circ} \frac{\rho^{(k)}}{\underline{\rho}^{(k)}} U^{(k)} \right] \\ & + \Delta_P \left[r^{(k)\circ} \frac{\rho^{(k)*}}{\underline{\rho}^{(k)}} \mathbf{HP} \left[\mathbf{u}^{(k)'} \right] \mathbf{S} \right] + \Delta_P \left[r^{(k)\circ} \frac{\rho^{(k)'}}{\underline{\rho}^{(k)}} \mathbf{u}^{(k)'} . \mathbf{S} \right] \end{aligned} \right\} \end{aligned} \tag{173}$$

Solving for volume fractions

The volume fractions can be obtained by solving the fluid continuity equations (Eq. (4)). However, if all the volume fractions are obtained by solving the continuity equations, the

geometric constraint will not be enforced unless the appropriate velocity field is available, which is not the case until convergence. One remedy is to solve for n-1 volume fractions using the fluid mass conservation equations, and use the geometric conservation equation to find the last volume fraction field. Thus, for fluids k=1 to n-1, solve

$$r_P^{(k)*} = H_P\left[r^{(k)*}\right. \tag{174}$$

and for k=n use

$$r_P^{(n)*} = 1 - \sum_{k=fluids \neq n} r_P^{(k)*} \tag{175}$$

A drawback of this procedure is that the volume fraction field of any phase is not influenced by the volume fraction fields of other phases except when calculating the n^{th} phase. This can affect the convergence rate negatively. Better solutions are presented below.

Improvement #3: Mutual influence of Volume Fractions

An improvement on the above procedure, is to solve all continuity equations to obtain all volume fractions and then enforce the geometric conservation constraint on the resolved volume fractions using the following equation:

$$r^{(k)} = \frac{r^{(k)*}}{\sum\limits_{m=1..n} r^{(m)*}} = \frac{H_P\left[r^{(k)*}\right]}{\sum\limits_{m=1..n} r^{(m)*}} \quad for \;\; k = 1....n \tag{176}$$

where $r^{(k)*}$ is the volume fraction resulting from the solution of the volume fraction equation for fluid k. The summation is carried out for all phases, and $r^{(k)}$ is the value of the volume fraction for fluid (k) which is carried into subsequent calculations. These $r^{(k)}$ of course, do sum to unity, and the solution of each of the volume fraction equations affect that of the remaining phases through the enforcement of the geometric constraint over all phase volume fractions rather than over the n^{th} phase only.

Improvement #4: Implicit Volume Fraction Equations

The solution of the volume fraction equations can be improved by implicitly accounting for the influence of the volume fractions of the different phases on each other. The details of the procedure will be presented for a two-fluid flow and then generalized to n-fluid flow situations. For that purpose, the following simplified form of the volume fraction equation is considered:

$$r_P^{(k)*} = H\left[r^{(k)*}\right] = \frac{\sum_{NB} A_{NB}^{(k)} r_{NB}^{(k)*} + B_P^{'(k)}}{A_P^{(k)}} \tag{177}$$

For the case of a two-fluid flow, the volume fraction equations can be written as

$$\begin{aligned} r_P^{(1)*} &= H_P\left[r^{(1)*}\right] = \frac{\sum_{NB} A_{NB}^{(1)} r_{NB}^{(1)*} + B_P^{'(1)}}{A_P^{(1)}} \\ r_P^{(2)*} &= H_P\left[r^{(2)*}\right] = \frac{\sum_{NB} A_{NB}^{(2)} r_{NB}^{(2)*} + B_P^{'(2)}}{A_P^{(2)}} \end{aligned} \tag{178}$$

The sum to 1 rule $\left(\sum_{phases} r^{(k)} = 1 \right)$ can be written in the following form

$$r_P^{(1)*} + r_P^{(2)*} = 1 = \frac{\sum_{NB} A_{NB}^{(1)} r_{NB}^{(1)*} + B_P^{'(1)}}{A_P^{(1)}} + \frac{\sum_{NB} A_{NB}^{(2)} r_{NB}^{(2)*} + B_P^{'(2)}}{A_P^{(2)}} \tag{179}$$

Based on this equation, one can write

$$A_P^{(1)} = \frac{A_P^{(2)}\left(\sum_{NB} A_{NB}^{(1)} r_{NB}^{(1)*} + B_P^{'(1)} \right) + A_P^{(1)}\left(\sum_{NB} A_{NB}^{(2)} r_{NB}^{(2)*} + B_P^{'(2)} \right)}{A_P^{(2)}} \tag{180}$$

Then, the volume fraction equation for fluid (1) can be rewritten as

$$r_P^{(1)*} = \frac{\left(\sum_{NB} A_{NB}^{(1)} r_{NB}^{(1)*} + B_P^{'(1)} \right)}{A_P^{(1)}\left(\frac{\sum_{NB} A_{NB}^{(1)} r_{NB}^{(1)*} + B_P^{'(1)}}{A_P^{(1)}} + \frac{\sum_{NB} A_{NB}^{(2)} r_{NB}^{(2)*} + B_P^{'(2)}}{A_P^{(2)}} \right)} \tag{181}$$

Equation (182) can be simplified into the form

$$r_P^{(1)*} = \frac{\left(\sum_{NB} A_{NB}^{(1)} r_{NB}^{(1)*} + B_P^{'(1)}\right)}{A_P^{(1)}\left(r_P^{(1)*} + \frac{\left(\sum_{NB} A_{NB}^{(1)} r_{NB}^{(1)*} + B_P^{'(1)}\right) - A_P^{(1)} r_P^{(1)*}}{A_P^{(1)}} + r_P^{(2)*} + \frac{\left(\sum_{NB} A_{NB}^{(2)} r_{NB}^{(2)*} + B_P^{'(2)}\right) - A_P^{(2)} r_P^{(2)*}}{A_P^{(2)}}\right)}$$

$$= \frac{\left(\sum_{NB} a_{NB}^{(1)} r_{NB}^{(1)*} + B_P^{'(1)}\right)}{A_P^{(1)}\left(1 - \frac{RESR^{(1)}}{A_P^{(1)}} - \frac{RESR^{(2)}}{A_P^{(2)}}\right)} \tag{182}$$

The equation for $r_P^{(2)*}$ can be derived in a similar manner.

For the case of n fluids, the kth volume fraction equation can easily be deduced from the above equation and is given by,

$$r_P^{(k)*} = \frac{\left(\sum_{NB} A_{NB}^{(k)} r_{NB}^{(k)*} + B_P^{'(k)}\right)}{A_P^{(k)}\left(1 - \sum_{m=\,phases} \frac{RESR^{(m)}}{A_P^{(m)}}\right)} \tag{183}$$

Improvement #5: Bounding the Volume Fractions

While the above techniques can be used to solve the volume fraction equations, it does not guarantee that the volume fraction values are bounded (i.e. between 0 and 1). This is a feature of iterative methods, which are known to return intermediate values that violate the set bounds. While these restrictions can be explicitly enforced after obtaining the solution from the discretized equations, a number of techniques were developed that lead to the implicit enforcing of these constraints.

The procedure developed by Carver [74] is based on a modification of the under-relaxation procedure of Patankar [16]. In the standard under-relaxation procedure a certain proportion of the original value is retained in the solution. Instead of solving the volume fraction

equation directly for $r^{(k)}$ to yield $r^{(k)imp}$, under-relaxation is used to yield a value $r^{(k)C}=\beta\, r^{(k)imp} +(1-\beta)\, r^{(k)\circ}$, where β is between [0,1] and $r^{(k)\circ}$ is the solution from the previous iteration. Relaxation may be imposed after the solution but this may detract from the integrity of the solution, and it is the norm to pre-relax the equation, making the relaxation method implied in the solution. Equation (25) is thus rewritten as:

$$\frac{A_P}{\beta} r_P^{(k)C} = \sum_{NB} A_{NB} r_{NB}^{(k)C} + B_P + \frac{(1-\beta)}{\beta} A_P r_P^{(k)\circ} \tag{184}$$

In the Carver procedure, the value of $r_P^{(k)C}$ over each control volume is monitored by explicitly calculating an intermediate value $r_P^{(k)\mathrm{int}}$ as:

$$r_P^{(k)\mathrm{int}} = \frac{\sum_{NB} A_{NB} r_{NB}^{(k)\circ} + B_P + \frac{(1-\beta)}{\beta} A_P r_P^{(k)\circ}}{\frac{A_P}{\beta}} \tag{185}$$

If $r_P^{(k)\mathrm{int}}>(1-\varepsilon)$ then the under-relaxation factor is modified to the form $\beta=\mathrm{MAX}(1-r_P^{(k)\circ},\gamma)$. The parameters ε and γ are small; values of 0.05 and 10^{-10} are suggested by Carver. The system of equations for each volume fraction is then solved implicitly using, for every control volume, the individually assigned relaxation parameter.

Solving the energy equations

The solutions of the energy equations follow that of the general multi-fluid scalar equation. As such, nothing new needs to be added in that regard (though in many cases coupling of the energy equation with the momentum and continuity equations is beneficial).

Part II - Geometric Conservation Based Algorithm

An example of a GCBA is the original IPSA, developed by Spalding [75], which introduces a stronger coupling between the pressure and the volume fractions. The sequence of events in the Geometric Conservation Based Algorithm (GCBA) is as follows:

- Solve the individual mass conservation equations for volume fractions.
- Solve the momentum equations for velocities.
- Solve the pressure correction equation.
- Correct velocity, volume fraction, density, and pressure fields.
- Solve the individual energy equations.
- Return to the first step and repeat until convergence.

As in the MCBA, the GCBA uses the momentum equations for a first estimate of velocities. However, the volume fractions are calculated without enforcing the geometric conservation equation. Hence, the mass conservation equations of all fluids are used to calculate the volume fractions. The pressure correction equation is based on the geometric conservation equation and is used to restore the imbalance of volume fractions. The errors in the calculated volume fractions are expressed in terms of pressure correction $(P'$, which is also used to adjust the velocity and density fields.

Solving for volume fractions

The solution starts by estimating the volume fractions using the continuity equations given as:

$$r_P^{(k)*} = H_P\left[r^{(k)*}\right] \quad for \;\; k = 1..n \;\; phases \tag{187}$$

There is nothing new to be added different than what was said earlier concerning the solution procedures used to solve the above set of equations and the previously described techniques are applicable.

Solving for velocities

Another important step in the solution cycle is to obtain an initial estimate for the velocity fields that satisfy the momentum equations. For that purpose, the following set of momentum equations (given here for completeness of presentation) are solved:

$$\mathbf{u}_P^{(k)} = \mathbf{H}_P\left[\mathbf{u}^{(k)}\right] - r_P^{(k)}\mathbf{D}_P^{(k)}\nabla P + \mathbf{D}_P^{(k)} \sum_{m=\ all\ fluids \neq k} g^{(km)}\mathbf{u}_P^{(m)} \tag{186}$$

Again, the same procedures used with the MCBA can be applied here including the PEA and SINCE techniques. Since these were explained earlier, they are not repeated here.

Solving for Pressure Correction

After solving the continuity equations for the volume fraction fields and the momentum equations for the velocity fields, the next step is to correct the various fields such that the volume fraction fields satisfy the compatibility equations and the velocity and pressure fields satisfy the continuity equations. For that purpose, an approach similar to the one used with the MCBA is adopted. The difference between the two approaches lies in the constraint equation employed in deriving the pressure or pressure correction equation. In the MCBA, the overall mass conservation equation was utilized. In the GCBA, the pressure correction equation is derived from the geometric conservation equation.

To start the derivation, it is noticed that initially the volume fraction field denoted by $r^{(k)*}$, does not satisfy the compatibility equation and a discrepancy exists i.e.

$$RESG_P = 1 - \sum_k r_P^{(k)*} \tag{188}$$

A change to $r^{(k)*}$ is sought that would restore the balance. The corrected r value, denoted by $r^{(k)}$ $\left(r^{(k)} = r^{(k)*} + r^{(k)'}\right)$, is such that

$$\sum_k \left(r^{(k)}\right) = \sum_k \left(r^{(k)*} + r^{(k)'}\right) = 1 \tag{189}$$

or

$$\sum_k \left(r^{(k)'}\right) = 1 - \sum_k \left(r^{(k)*}\right) = RESG_P \tag{190}$$

Correction to the volume fraction, $r^{(k)'}$, will be associated with a correction to the velocity, density, and pressure fields, $\mathbf{u}^{(k)'}$, $\rho^{(k)'}$, and P' respectively. Thus, the corrected fields are given as:

$$\begin{cases} r^{(k)} = r^{(k)*} + r^{(k)'} \\ P = P^{\circ} + P' \\ \mathbf{u}^{(k)} = \mathbf{u}^{(k)*} + \mathbf{u}^{(k)'} \qquad \left(u^{(k)} = u^{(k)*} + u^{(k)'}, v^{(k)} = v^{(k)*} + v^{(k)'}, w^{(k)} = w^{(k)*} + w^{(k)'}\right. \\ \rho^{(k)} = \rho^{(k)\circ} + \rho^{(k)'} \end{cases} \tag{191}$$

Recalling that the corrected continuity equation of phase (k) can be written in discretized form as

$$\frac{\left(r_P^{(k)}\rho_P^{(k)}\right) - \left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega_P + \Delta_P\left[r^{(k)}\rho^{(k)}\mathbf{u}^{(k)}.\mathbf{S}\right] = \acute{M}_P^{(k)} r_P^{(k)}\Omega_P \tag{192}$$

Its expanded form becomes:

$$\begin{aligned} &\frac{\left(r_P^{(k)*} + r_P^{(k)'}\right)\left(\rho_P^{(k)\circ} + \rho_P^{(k)'}\right) - \left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega_P \\ &\quad + \Delta_P\left(\left(r^{(k)*} + r^{(k)'}\right)\left(\rho^{(k)\circ} + \rho^{(k)'}\right)\left(\mathbf{u}^{(k)*} + \mathbf{u}^{(k)'}\right)\mathbf{S}\right) = \acute{M}_P^{(k)}\left(r_P^{(k)*} + r_P^{(k)'}\right)\Omega_P \end{aligned} \tag{193}$$

or

$$\frac{\left(r_P^{(k)*}\rho_P^{(k)\circ}+r_P^{(k)*}\rho_P^{(k)'}+r_P^{(k)'}\rho_P^{(k)\circ}+r_P^{(k)'}\rho_P^{(k)'}\right)-\left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega_P+$$
$$\Delta_P\left[\begin{pmatrix} r^{(k)*}\rho^{(k)\circ}\mathbf{u}^{(k)*}+r^{(k)*}\rho^{(k)\circ}\mathbf{u}^{(k)'}+r^{(k)*}\rho^{(k)'}\mathbf{u}^{(k)*}+\\ r^{(k)*}\rho^{(k)'}\mathbf{u}^{(k)'}+r^{(k)'}\rho^{(k)\circ}\mathbf{u}^{(k)*}+r^{(k)'}\rho^{(k)\circ}\mathbf{u}^{(k)'}+\\ r^{(k)'}\rho^{(k)'}\mathbf{u}^{(k)*}+r^{(k)'}\rho^{(k)'}\mathbf{u}^{(k)'}\end{pmatrix}.\mathbf{S}\right]=\dot{M}_P^{(k)}r_P^{(k)*}\Omega_P+\dot{M}_P^{(k)}r_P^{(k)'}\Omega_P \tag{194}$$

Further expansion and rearrangement of the above equation yields

$$\begin{aligned}&\frac{\left(r_P^{(k)*}\rho_P^{(k)'}+r_P^{(k)'}\rho_P^{(k)\circ}\right)}{\delta t}\Omega_P+\Delta_P\left[\left(r^{(k)*}\rho^{(k)\circ}\mathbf{u}^{(k)'}.\mathbf{S}+r^{(k)*}U^{(k)*}\rho_P^{(k)'}+\rho^{(k)\circ}U^{(k)*}r^{(k)'}\right)\right]\\&-\dot{M}_P^{(k)}r_P^{(k)'}\Omega_P=-\frac{\left(r_P^{(k)*}\rho_P^{(k)\circ}\right)-\left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega_P-\Delta_P\left[\left(r^{(k)*}\rho^{(k)\circ}U^{(k)*}\right)\right]\\&-\frac{\left(r_P^{(k)'}\rho_P^{(k)'}\right)}{\delta t}\Omega_P-\Delta_P\left[\begin{pmatrix} r^{(k)*}\rho_P^{(k)'}\mathbf{u}^{(k)'}+r_P^{(k)'}\rho^{(k)\circ}\mathbf{u}^{(k)'}+\\ r_P^{(k)'}\rho_P^{(k)'}\mathbf{u}^{(k)*}+r_P^{(k)'}\rho_P^{(k)'}\mathbf{u}^{(k)'}\end{pmatrix}.\mathbf{S}\right]+\dot{M}_P^{(k)}r_P^{(k)*}\Omega_P\end{aligned} \tag{195}$$

Neglecting second and third order terms (i.e. $r_P^{(k)'}\rho_P^{(k)'}$, $\rho_P^{(k)'}\mathbf{u}^{(k)'}$, $r_P^{(k)'}\mathbf{u}^{(k)'}$, and $r_P^{(k)'}\rho_P^{(k)'}\mathbf{u}^{(k)'}$) the above equation reduces to:

$$\begin{aligned}&\frac{\left(r_P^{(k)*}\rho_P^{(k)'}+r_P^{(k)'}\rho_P^{(k)\circ}\right)}{\delta t}\Omega_P+\Delta_P\left[\left(r^{(k)*}\rho^{(k)\circ}\mathbf{u}^{(k)'}.\mathbf{S}+r^{(k)*}U^{(k)*}\rho_P^{(k)'}+\rho^{(k)\circ}U^{(k)*}r^{(k)'}\right)\right]\\&-\dot{M}_P^{(k)}r_P^{(k)'}\Omega_P=-\frac{\left(r_P^{(k)*}\rho_P^{(k)\circ}\right)-\left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega_P-\Delta_P\left[\left(r^{(k)*}\rho^{(k)\circ}U^{(k)*}\right)\right]+\dot{M}_P^{(k)}r_P^{(k)*}\Omega_P\end{aligned} \tag{196}$$

Writing $\mathbf{u}^{(k)'}$ as a function of P', similar to what is usually done in a simple-like algorithm, the correction momentum equations become

$$\mathbf{u}^{(k)'}=\mathbf{HP}[\mathbf{u}^{(k)'}]-r^{(k)*}\mathbf{D}^{(k)}\nabla P'-r^{(k)'}\mathbf{D}^{(k)}\nabla P^{\circ}-r^{(k)'}\mathbf{D}^{(k)}\nabla P' \tag{197}$$

Substituting and rearranging, one gets

$$\begin{aligned}&\frac{\left(\rho_P^{(k)\circ}\Omega_P\right)}{\delta t}r_P^{(k)'}+\Delta_P\left[\left\{\rho^{(k)\circ}U^{(k)*}-r^{(k)*}\rho^{(k)\circ}\left(\mathbf{D}^{(k)}\nabla P^{\circ}\right)\mathbf{S}\right\}r^{(k)'}\right]-\dot{M}_P^{(k)}r_P^{(k)'}\Omega_P=\\&-\frac{\left(r_P^{(k)*}\Omega_P\right)}{\delta t}\rho_P^{(k)'}-\Delta_P\left[r^{(k)*}\rho^{(k)\circ}\left(\mathbf{HP}[\mathbf{u}^{(k)'}]-r^{(k)*}\mathbf{D}^{(k)}\nabla P'-r^{(k)'}\mathbf{D}^{(k)}\nabla P'\right)\mathbf{S}+r^{(k)*}U^{(k)*}\rho^{(k)'}\right.\\&-\frac{\left(r_P^{(k)*}\rho_P^{(k)\circ}\right)-\left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega_P-\Delta_P\left[\left(r^{(k)*}\rho^{(k)\circ}U^{(k)*}\right)\right]+\dot{M}_P^{(k)}r_P^{(k)*}\Omega_P\end{aligned} \tag{198}$$

Upon discretizing the left-hand side of the above equation, it can be rewritten as

$$A_P^{(k)} r_P^{(k)'} + \sum_{NB=E,W,N,S,T,B} A_{NB}^{(k)} r_{NB}^{(k)'} = \\ -\frac{\left(r_P^{(k)*}\Omega_P\right)}{\delta t}\rho_P^{(k)'} - \Delta_P\left[r^{(k)*}\rho^{(k)\circ}\begin{pmatrix}\mathbf{HP}[\mathbf{u}^{(k)'}] - r^{(k)*}\mathbf{D}^{(k)}\nabla P' \\ -r^{(k)'}\mathbf{D}^{(k)}\nabla P'\end{pmatrix}.\mathbf{S} + r^{(k)*}U^{(k)*}\rho^{(k)'}\right] \\ -\frac{\left(r_P^{(k)*}\rho_P^{(k)\circ}\right) - \left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega_P - \Delta_P\left[\left(r^{(k)*}\rho^{(k)\circ}U^{(k)*}\right)\right] + \dot{M}_P^{(k)} r_P^{(k)*}\Omega_P \tag{199}$$

By noticing that the left-hand side of Eq. (199) resembles the volume fraction equation (Eq. 39), it can be written symbolically as

$$r_P^{(k)'} - H_P\left[r^{(k)'}\right] = \\ -R_P^{(k)}\left(\begin{aligned}&\frac{r_P^{(k)*}\Omega_P}{\delta t}\rho_P^{(k)'} + \Delta_P\left[r^{(k)*}\rho^{(k)\circ}\begin{pmatrix}\mathbf{HP}[\mathbf{u}^{(k)'}] - r^{(k)*}\mathbf{D}^{(k)}\nabla P' \\ -r^{(k)'}\mathbf{D}^{(k)}\nabla P'\end{pmatrix}.\mathbf{S} + r^{(k)*}U^{(k)*}\rho^{(k)'}\right] \\ &+\frac{\left(r_P^{(k)*}\rho_P^{(k)\circ}\right) - \left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega_P + \Delta_P\left[\left(r^{(k)*}\rho^{(k)\circ}U^{(k)*}\right)\right] - \dot{M}_P^{(k)} r_P^{(k)*}\Omega_P\end{aligned}\right) \tag{200}$$

where $R_P^{(k)} = 1/A_P^{(k)}$.

Neglecting the correction to neighboring cells, equation (200) reduces to:

$$r_P^{(k)'} = -R_P^{(k)}\left(\begin{aligned}&\frac{r_P^{(k)*}\Omega_P}{\delta t}\rho_P^{(k)'} + \Delta_P\left[r^{(k)*}\rho^{(k)\circ}\begin{pmatrix}\mathbf{HP}[\mathbf{u}^{(k)'}] - r^{(k)*}\mathbf{D}^{(k)}\nabla P' \\ -r^{(k)'}\mathbf{D}^{(k)}\nabla P'\end{pmatrix}.\mathbf{S} + r^{(k)*}U^{(k)*}\rho^{(k)'}\right] \\ &+\frac{\left(r_P^{(k)*}\rho_P^{(k)\circ}\right) - \left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega_P + \Delta_P\left[\left(r^{(k)*}\rho^{(k)\circ}U^{(k)*}\right)\right] - \dot{M}_P^{(k)} r_P^{(k)*}\Omega_P\end{aligned}\right) \tag{201}$$

Substitution of this equation into the geometric conservation equation yields

$$\sum_k\left\{-R_P^{(k)}\left(\begin{aligned}&\frac{r_P^{(k)*}\Omega_P}{\delta t}\rho_P^{(k)'} + \Delta_P\left[\begin{aligned}&r^{(k)*}\rho^{(k)\circ}\begin{pmatrix}\mathbf{HP}[\mathbf{u}^{(k)'}] - r^{(k)*}\mathbf{D}^{(k)}\nabla P' \\ -r^{(k)'}\mathbf{D}^{(k)}\nabla P'\end{pmatrix}.\mathbf{S} \\ &+r^{(k)*}U^{(k)*}\rho^{(k)'}\end{aligned}\right] \\ &+\frac{\left(r_P^{(k)*}\rho_P^{(k)\circ}\right) - \left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega_P + \Delta_P\left[\left(r^{(k)*}\rho^{(k)\circ}U^{(k)*}\right)\right]\end{aligned}\right)\right\} = RESG_P \tag{202}$$

Writing density correction in terms of pressure correction (i.e. $\rho^{(k)'} = C_\rho^{(k)}P'$), equation (202) is transformed to

$$\sum_k \left\{ -R_P^{(k)} \left(\begin{array}{l} \dfrac{r_P^{(k)*}\Omega_P C_\rho^{(k)}}{\delta t} P'_P + \Delta_P\left[r^{(k)*}U^{(k)*}C_\rho^{(k)}P'\right] + \\ \Delta_P\left[r^{(k)*}\rho^{(k)\circ}\left(\mathbf{HP}[\mathbf{u}^{(k)'}] - r^{(k)*}\mathbf{D}^{(k)}\nabla P' - r^{(k)'}\mathbf{D}^{(k)}\nabla P'\right)\mathbf{S}\right] \\ + \dfrac{\left(r_P^{(k)*}\rho_P^{(k)\circ}\right) - \left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega_P + \Delta_P\left[\left(r^{(k)*}\rho^{(k)\circ}U^{(k)*}\right)\right] \end{array} \right) \right\} = RESG_P \tag{203}$$

As detailed next, the above equation can be expanded using any simple-like algorithm to yield a new family of multi-fluid flow algorithms (GCBA-SIMPLE, GCBA-SIMPLEC, GCBA-PISO,…).

The GCBA following SIMPLE (GCBA-SIMPLE): Symbolic Form

<u>Predictor:</u>

$$r_P^{(k)*} = H_P[r^{(k)*}] \tag{204}$$

$$\mathbf{u}_P^{(k)*} = \mathbf{HP}_P[\mathbf{u}^{(k)*}] - r_P^{(k)*}\mathbf{D}_P^{(k)}\nabla_P P^{\circ} \tag{205}$$

<u>Corrector:</u>

$$\left(\mathbf{u}^{(k)'}, P', \rho^{(k)'}, r^{(k)'}\right)\left(\begin{array}{l} \mathbf{u}^{(k)**} = \mathbf{u}^{(k)*} + \mathbf{u}^{(k)'}, P^{*} = P^{\circ} + P', \\ \rho^{(k)*} = \rho^{(k)\circ} + \rho^{(k)'}, r^{(k)**} = r^{(k)*} + r^{(k)'} \end{array} \right) \tag{206}$$

$$\therefore \mathbf{u}_P^{(k)**} = \mathbf{HP}_P[\mathbf{u}^{(k)**}] - r_P^{(k)**}\mathbf{D}_P^{(k)}\nabla_P P^{*} = \mathbf{HP}_P[\mathbf{u}^{(k)*} + \mathbf{u}^{(k)'}] - \left(r_P^{(k)*} + r_P^{(k)'}\right)\mathbf{D}_P^{(k)}\nabla_P\left(P^{\circ} + P'\right. \tag{207}$$

$$\therefore \left\{ \begin{array}{l} \mathbf{u}_P^{(k)'} = \mathbf{HP}_P[\mathbf{u}^{(k)'}] - r_P^{(k)*}\mathbf{D}_P^{(k)}\nabla_P P' - r_P^{(k)'}\mathbf{D}_P^{(k)}\nabla_P P^{\circ} - r_P^{(k)'}\mathbf{D}_P^{(k)}\nabla_P P' \\ \rho^{(k)'} = C_\rho^{(k)} P' \\ r_P^{(k)'} = -R_P^{(k)}\left(\dfrac{r_P^{(k)*}\Omega_P}{\delta t}\rho_P^{(k)'} + \Delta_P\left[\left(r^{(k)*}\rho^{(k)\circ}\mathbf{u}^{(k)'}\right)\mathbf{S} + r^{(k)*}U^{(k)*}\rho^{(k)'}\right]\right) \end{array} \right. \tag{208}$$

<u>Condition:</u>

$$\sum_k \left\{r_P^{(k)'}\right\} = RESG_P \tag{209}$$

$$\therefore \sum_k \left\{ -R_P^{(k)} \left(\begin{array}{l} \dfrac{r_P^{(k)*}\Omega_P C_\rho^{(k)}}{\delta t} P'_P + \Delta_P\left[r^{(k)*}U^{(k)*}C_\rho^{(k)}P'\right] + \\ \Delta_P\left[r^{(k)*}\rho^{(k)\circ}\left(\mathbf{HP}[\mathbf{u}^{(k)'}] - r^{(k)*}\mathbf{D}^{(k)}\nabla P' - r^{(k)'}\mathbf{D}^{(k)}\nabla P'\right).\mathbf{S}\right] \\ + \dfrac{\left(r_P^{(k)*}\rho_P^{(k)\circ}\right) - \left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega_P + \Delta_P\left[\left(r^{(k)*}\rho^{(k)\circ}U^{(k)*}\right)\right] \end{array} \right) \right\} = RESG_P \tag{210}$$

$$\begin{array}{l} \displaystyle\sum_k \left\{ R_P^{(k)} \left(\frac{r_P^{(k)*}\Omega_P C_\rho^{(k)}}{\delta t} P'_P + \Delta_P\left[r^{(k)*}U^{(k)*}C_\rho^{(k)}P'\right] - \Delta_P\left[r^{(k)*}\rho^{(k)\circ}\left(r^{(k)*}\mathbf{D}^{(k)}\nabla P'\right).\mathbf{S}\right] \right) \right\} = \\ \therefore \quad -\displaystyle\sum_k \left\{ R_P^{(k)} \left(\begin{array}{l} \dfrac{\left(r_P^{(k)*}\rho_P^{(k)\circ}\right) - \left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega_P + \Delta_P\left[\left(r^{(k)*}\rho^{(k)\circ}U^{(k)*}\right)\right] + \\ \Delta_P\left[r^{(k)*}\rho^{(k)\circ}\left(\mathbf{HP}[\mathbf{u}^{(k)'}] - r^{(k)'}\mathbf{D}^{(k)}\nabla P'\right).\mathbf{S}\right] \end{array} \right) \right\} - RESG_P \end{array} \tag{211}$$

Approximation:

Neglect: $\mathbf{HP}[\mathbf{u}^{(k)'}]$, $r^{(k)'}\mathbf{D}^{(k)}\nabla P'$

$$\Rightarrow \mathbf{u}_P^{(k)'} = -r_P^{(k)*}\mathbf{D}_P^{(k)}\nabla_P P' \tag{212}$$

Approximate Equation:

$$\begin{array}{l} \displaystyle\sum_k \left\{ R_P^{(k)} \left(\frac{r_P^{(k)*}\Omega_P C_\rho^{(k)}}{\delta t} P'_P + \Delta_P\left[r^{(k)*}U^{(k)*}C_\rho^{(k)}P'\right] - \Delta_P\left[r^{(k)*}\rho^{(k)\circ}\left(r^{(k)*}\mathbf{D}^{(k)}\nabla P'\right).\mathbf{S}\right] \right) \right\} = \\ \Rightarrow \quad -\displaystyle\sum_k \left\{ R_P^{(k)} \left(\frac{\left(r_P^{(k)*}\rho_P^{(k)\circ}\right) - \left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega_P + \Delta_P\left[\left(r^{(k)*}\rho^{(k)\circ}U^{(k)*}\right)\right] \right) \right\} - RESG_P \end{array} \tag{213}$$

A Global GCBA-SIMPLE Iteration

- *Solve implicitly for the volume fraction fields.*
- *Solve implicitly for* $\mathbf{u}^{(k)}$*, using the old pressure, density, and volume fraction fields.*
- *Calculate the* $\mathbf{D}^{(k)}$ *fields.*
- *Solve the pressure correction equation.*
- *Correct* $\mathbf{u}^{(k)}$, P, $r^{(k)}$ *and* $\rho^{(k)}$.
- *Solve implicitly the energy equations and update the density fields.*
- *Return to the first step and iterate until convergence.*

The GCBA following SIMPLEC (GCBA-SIMPLEC): Symbolic Form

Predictor:

$$r_P^{(k)*} = H_P[r^{(k)*}] \tag{214}$$

$$\mathbf{u}_P^{(k)*} = \mathbf{HP}_P[\mathbf{u}^{(k)*}] - r_P^{(k)*}\mathbf{D}_P^{(k)}\nabla_P P^\circ \tag{215}$$

Corrector:

$$\left(\mathbf{u}^{(k)'}, P', \rho^{(k)'}, r^{(k)'}\right)\begin{pmatrix}\mathbf{u}^{(k)**} = \mathbf{u}^{(k)*} + \mathbf{u}^{(k)'}, P^* = P^\circ + P', \\ \rho^{(k)*} = \rho^{(k)\circ} + \rho^{(k)'}, r^{(k)**} = r^{(k)*} + r^{(k)'}\end{pmatrix} \tag{216}$$

$$\therefore \mathbf{u}_P^{(k)**} = \mathbf{HP}_P[\mathbf{u}^{(k)**}] - r_P^{(k)**}\mathbf{D}_P^{(k)}\nabla_P P^* = \mathbf{HP}_P[\mathbf{u}^{(k)*} + \mathbf{u}^{(k)'}] - \left(r_P^{(k)*} + r_P^{(k)'}\right)\mathbf{D}_P^{(k)}\nabla_P\left(P^\circ + P'\right. \tag{217}$$

$$\therefore \mathbf{u}_P^{(k)'} = \mathbf{HP}_P[\mathbf{u}^{(k)'}] - r_P^{(k)*}\mathbf{D}_P^{(k)}\nabla_P P' - r_P^{(k)'}\mathbf{D}_P^{(k)}\nabla_P P^\circ - r_P^{(k)'}\mathbf{D}_P^{(k)}\nabla_P P' \tag{218}$$

Subtracting $\tilde{\mathbf{HP}}_P[1]\mathbf{u}_P^{(k)'}$ from both sides, one gets

$$\mathbf{u}_P^{(k)'} - \tilde{\mathbf{HP}}_P[1]\mathbf{u}_P^{(k)'} = \begin{pmatrix}\mathbf{HP}_P[\mathbf{u}^{(k)'}] - \tilde{\mathbf{HP}}_P[1]\mathbf{u}_P^{(k)'} - r_P^{(k)*}\mathbf{D}_P^{(k)}\nabla_P P' \\ -r_P^{(k)'}\mathbf{D}_P^{(k)}\nabla_P P^\circ - r_P^{(k)'}\mathbf{D}_P^{(k)}\nabla_P P'\end{pmatrix} \tag{219}$$

$$\therefore \left(1 - \tilde{\mathbf{HP}}_P[1]\right)\mathbf{u}_P^{(k)'} = \mathbf{HP}_P[\mathbf{u}^{(k)'} - \mathbf{u}_P^{(k)'}] - r_P^{(k)*}\mathbf{D}_P^{(k)}\nabla_P P' - r_P^{(k)'}\mathbf{D}_P^{(k)}\nabla_P P^\circ - r_P^{(k)'}\mathbf{D}_P^{(k)}\nabla_P P' \tag{220}$$

$$\therefore \mathbf{u}_P^{(k)'} = \begin{bmatrix}\dfrac{\mathbf{HP}_P[\mathbf{u}^{(k)'} - \mathbf{u}_P^{(k)'}]}{\left(1 - \tilde{\mathbf{HP}}_P[1]\right)} - r_P^{(k)*}\dfrac{\mathbf{D}_P^{(k)}}{\left(1 - \tilde{\mathbf{HP}}_P[1]\right)}\nabla_P P' \\ -r_P^{(k)'}\dfrac{\mathbf{D}_P^{(k)}}{\left(1 - \tilde{\mathbf{HP}}_P[1]\right)}\nabla_P P^\circ - r_P^{(k)'}\dfrac{\mathbf{D}_P^{(k)}}{\left(1 - \tilde{\mathbf{HP}}_P[1]\right)}\nabla_P P'\end{bmatrix} \tag{221}$$

$$\therefore \begin{cases}\mathbf{u}_P^{(k)'} = \dfrac{\mathbf{HP}_P[\mathbf{u}^{(k)'} - \mathbf{u}_P^{(k)'}]}{\left(1 - \tilde{\mathbf{HP}}_P[1]\right)} - r_P^{(k)*}\tilde{\mathbf{D}}_P^{(k)}\nabla_P P' - r_P^{(k)'}\tilde{\mathbf{D}}_P^{(k)}\nabla_P P^\circ - r_P^{(k)'}\tilde{\mathbf{D}}_P^{(k)}\nabla_P P' \\ \rho^{(k)'} = C_\rho^{(k)}P' \\ r_P^{(k)'} = -R_P^{(k)}\left(\dfrac{r_P^{(k)*}\Omega_P}{\delta t}\rho_P^{(k)'} + \Delta_P\left[\left(r^{(k)*}\rho^{(k)\circ}\mathbf{u}^{(k)'}\right)\mathbf{S} + r^{(k)*}U^{(k)*}\rho^{(k)'}\right]\right)\end{cases} \tag{222}$$

Condition:

$$\sum_k \left\{r_P^{(k)'}\right\} = RESG_P \tag{223}$$

$$\therefore \sum_k \left\{ -R_P^{(k)} \left(\begin{array}{l} \dfrac{r_P^{(k)*}\Omega_P C_\rho^{(k)}}{\delta t} P'_P + \Delta_P\left[r^{(k)*}U^{(k)*}C_\rho^{(k)}P'\right] + \\ \Delta_P\left[r^{(k)*}\rho^{(k)\circ}\left(\dfrac{\mathbf{HP}[\mathbf{u}^{(k)'} - \mathbf{u}_P^{(k)'}]}{\left(1-\tilde{\mathbf{HP}}_P[1]\right)} - r^{(k)*}\tilde{\mathbf{D}}^{(k)}\nabla P' - r^{(k)'}\tilde{\mathbf{D}}^{(k)}\nabla P' \right).\mathbf{S} \right] \\ + \dfrac{\left(r_P^{(k)*}\rho_P^{(k)\circ}\right) - \left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega_P + \Delta_P\left[\left(r^{(k)*}\rho^{(k)\circ}U^{(k)*}\right)\right] \end{array} \right) \right\} = RESG_P \quad (224)$$

$$\begin{array}{l} \sum_k \left\{ R_P^{(k)}\left(\dfrac{r_P^{(k)*}\Omega_P C_\rho^{(k)}}{\delta t} P'_P + \Delta_P\left[r^{(k)*}U^{(k)*}C_\rho^{(k)}P'\right] - \Delta_P\left[r^{(k)*}\rho^{(k)\circ}\left(r^{(k)*}\tilde{\mathbf{D}}^{(k)}\nabla P'\right)\mathbf{S}\right] \right) \right\} = \\ -\sum_k \left\{ R_P^{(k)}\left(\begin{array}{l} \dfrac{\left(r_P^{(k)*}\rho_P^{(k)\circ}\right) - \left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega_P + \Delta_P\left[\left(r^{(k)*}\rho^{(k)\circ}U^{(k)*}\right)\right] + \\ \Delta_P\left[r^{(k)*}\rho^{(k)\circ}\left(\mathbf{HP}[\mathbf{u}^{(k)'} - \mathbf{u}_P^{(k)'}] - r^{(k)'}\tilde{\mathbf{D}}^{(k)}\nabla P'\right)\mathbf{S}\right] \end{array} \right) \right\} - RESG_P \end{array} \quad (225)$$

<u>Approximation:</u>

Neglect: $\mathbf{HP}[\mathbf{u}^{(k)'} - \mathbf{u}_P^{(k)'}]$, $r^{(k)'}\tilde{\mathbf{D}}^{(k)}\nabla P'$

$$\Rightarrow \mathbf{u}_P^{(k)'} = -r_P^{(k)*}\mathbf{D}_P^{(k)}\nabla_P P' \quad (226)$$

<u>Approximate Equation:</u>

$$\Rightarrow \begin{array}{l} \sum_k \left\{ R_P^{(k)}\left(\dfrac{r_P^{(k)*}\Omega_P C_\rho^{(k)}}{\delta t} P'_P + \Delta_P\left[r^{(k)*}U^{(k)*}C_\rho^{(k)}P'\right] - \Delta_P\left[r^{(k)*}\rho^{(k)\circ}\left(r^{(k)*}\tilde{\mathbf{D}}^{(k)}\nabla P'\right)\mathbf{S}\right] \right) \right\} = \\ -\sum_k \left\{ R_P^{(k)}\left(\dfrac{\left(r_P^{(k)*}\rho_P^{(k)\circ}\right) - \left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega_P + \Delta_P\left[\left(r^{(k)*}\rho^{(k)\circ}U^{(k)*}\right)\right] \right) \right\} - RESG_P \end{array} \quad (227)$$

A Global GCBA-SIMPLEC Iteration

- *Solve implicitly for the volume fraction fields.*
- *Solve implicitly for* $\mathbf{u}^{(k)}$*, using the old pressure, density, and volume fraction fields.*
- *Calculate the* $\tilde{\mathbf{D}}^{(k)}$ *fields.*
- *Solve the pressure correction equation.*
- *Correct* $\mathbf{u}^{(k)}$, P, $r^{(k)}$ *and* $\rho^{(k)}$.
- *Solve implicitly the energy equations and update the density fields.*
- *Return to the first step and iterate until convergence.*

The GCBA following PRIME (GCBA-PRIME): Symbolic Form

Predictor:

$$r_P^{(k)*} = H_P[r^{(k)\circ}] \tag{228}$$

$$\mathbf{u}_P^{(k)*} = \mathbf{HP}_P[\mathbf{u}^{(k)\circ}] - r_P^{(k)*}\mathbf{D}_P^{(k)}\nabla_P P^\circ \tag{229}$$

Corrector:

$$\left(\mathbf{u}^{(k)'}, P', \rho^{(k)'}, r^{(k)'}\right)\left(\begin{matrix}\mathbf{u}^{(k)**} = \mathbf{u}^{(k)*} + \mathbf{u}^{(k)'}, P^* = P^\circ + P', \\ \rho^{(k)*} = \rho^{(k)\circ} + \rho^{(k)'}, r^{(k)**} = r^{(k)*} + r^{(k)'}\end{matrix}\right) \tag{230}$$

$$\therefore \mathbf{u}_P^{(k)**} = \mathbf{HP}_P[\mathbf{u}^{(k)**}] - r_P^{(k)**}\mathbf{D}_P^{(k)}\nabla_P P^* = \mathbf{HP}_P[\mathbf{u}^{(k)*} + \mathbf{u}^{(k)'}] - \left(r_P^{(k)*} + r_P^{(k)'}\right)\mathbf{D}_P^{(k)}\nabla_P\left(P^\circ + P'\right) \tag{231}$$

$$\therefore \begin{cases} \mathbf{u}_P^{(k)'} = \mathbf{HP}_P[\mathbf{u}^{(k)*} - \mathbf{u}^{(k)\circ}] + \mathbf{HP}_P[\mathbf{u}^{(k)'}] - r_P^{(k)*}\mathbf{D}_P^{(k)}\nabla_P P' - r_P^{(k)'}\mathbf{D}_P^{(k)}\nabla_P P^\circ - r_P^{(k)'}\mathbf{D}_P^{(k)}\nabla_P P' \\ \rho^{(k)'} = C_\rho^{(k)} P' \\ r_P^{(k)'} = -R_P^{(k)}\left(\dfrac{r_P^{(k)*}\Omega_P}{\delta t}\rho_P^{(k)'} + \Delta_P\left[\left(r^{(k)*}\rho^{(k)\circ}\mathbf{u}^{(k)'}\right)\mathbf{S} + r^{(k)*}U^{(k)*}\rho^{(k)'}\right]\right) \end{cases} \tag{232}$$

Condition:

$$\sum_k \left\{r_P^{(k)'}\right\} = RESG_P \tag{233}$$

$$\therefore \sum_k \left\{ -R_P^{(k)} \left(\begin{matrix} \dfrac{r_P^{(k)*}\Omega_P C_\rho^{(k)}}{\delta t} P_P' + \Delta_P\left[r^{(k)*}U^{(k)*}C_\rho^{(k)}P'\right] + \\ \Delta_P\left[r^{(k)*}\rho^{(k)\circ}\left(\begin{matrix}\mathbf{HP}[\mathbf{u}^{(k)*} - \mathbf{u}^{(k)\circ}] + \mathbf{HP}[\mathbf{u}^{(k)'}] \\ -r^{(k)*}\mathbf{D}^{(k)}\nabla P' - r^{(k)'}\mathbf{D}^{(k)}\nabla P'\end{matrix}\right).\mathbf{S}\right] \\ + \dfrac{\left(r_P^{(k)*}\rho_P^{(k)\circ}\right) - \left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega_P + \Delta_P\left[\left(r^{(k)*}\rho^{(k)\circ}U^{(k)*}\right)\right] \end{matrix} \right) \right\} = RESG_P \tag{234}$$

$$\begin{aligned} &\sum_k \left\{ R_P^{(k)}\left(\frac{r_P^{(k)*}\Omega_P C_\rho^{(k)}}{\delta t} P_P' + \Delta_P\left[r^{(k)*}U^{(k)*}C_\rho^{(k)}P'\right] - \Delta_P\left[r^{(k)*}\rho^{(k)\circ}\left(r^{(k)*}\mathbf{D}^{(k)}\nabla P'\right)\mathbf{S}\right] \right) \right\} = \\ \therefore \; &-\sum_k \left\{ R_P^{(k)} \left(\begin{matrix} \dfrac{\left(r_P^{(k)*}\rho_P^{(k)\circ}\right) - \left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega_P + \Delta_P\left[\left(r^{(k)*}\rho^{(k)\circ}U^{(k)*}\right)\right] + \\ \Delta_P\left[r^{(k)*}\rho^{(k)\circ}\left(\mathbf{HP}[\mathbf{u}^{(k)*} - \mathbf{u}^{(k)\circ}] + \mathbf{HP}[\mathbf{u}^{(k)'}] - r^{(k)'}\mathbf{D}^{(k)}\nabla P'\right)\mathbf{S}\right] \end{matrix} \right) \right\} - RESG_P \end{aligned} \tag{235}$$

Approximation:

Neglect: $\mathbf{HP}[\mathbf{u}^{(k)*}-\mathbf{u}^{(k)\circ}]$, $\mathbf{HP}[\mathbf{u}^{(k)'}]$, $r^{(k)'}\mathbf{D}^{(k)}\nabla P'$

$$\Rightarrow \mathbf{u}_P^{(k)'} = -r_P^{(k)*}\mathbf{D}_P^{(k)}\nabla_P P' \tag{236}$$

Approximate Equation:

$$\Rightarrow \begin{aligned} &\sum_k \left\{ R_P^{(k)}\left(\frac{r_P^{(k)*}\Omega_P C_\rho^{(k)}}{\delta t} P_P' + \Delta_P\left[r^{(k)*}U^{(k)*}C_\rho^{(k)}P'\right] - \Delta_P\left[r^{(k)*}\rho^{(k)\circ}\left(r^{(k)*}\mathbf{D}^{(k)}\nabla P'\right).\mathbf{S}\right]\right)\right\} = \\ &-\sum_k \left\{ R_P^{(k)}\left(\frac{\left(r_P^{(k)*}\rho_P^{(k)\circ}\right) - \left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega_P + \Delta_P\left[\left(r^{(k)*}\rho^{(k)\circ}U^{(k)*}\right)\right]\right)\right\} - RESG_P \end{aligned} \tag{237}$$

A Global GCBA-PRIME Iteration

- *Solve explicitly for the volume fraction fields.*
- *Solve explicitly for* $\mathbf{u}^{(k)}$*, using the old pressure and density fields.*
- *Calculate the* $\mathbf{D}^{(k)}$ *fields.*
- *Solve the pressure correction equation.*
- *Correct* $\mathbf{u}^{(k)}$, P, $r^{(k)}$ *and* $\rho^{(k)}$*.*
- *Solve implicitly the energy equations and update the density fields.*
- *Return to the first step and iterate until convergence.*

The GCBA following SIMPLEST (GCBA-SIMPLEST): Symbolic Form

Predictor:

$$r_P^{(k)*} = H_P^D[r^{(k)*}] + H_P^C[r^{(k)\circ}] \tag{238}$$

$$\mathbf{u}_P^{(k)*} = \mathbf{HP}_P^D\left[\mathbf{u}^{(k)*}\right] + \mathbf{HP}_P^C\left[\mathbf{u}^{(k)\circ}\right] - r_P^{(k)*}\mathrm{D}_P^{(k)}\nabla_P P^\circ \tag{239}$$

Corrector:

$$\left(\mathbf{u}^{(k)'}, P', \rho^{(k)'}, r^{(k)'}\right)\left(\begin{aligned} &\mathbf{u}^{(k)**} = \mathbf{u}^{(k)*} + \mathbf{u}^{(k)'}, P^* = P^\circ + P', \\ &\rho^{(k)*} = \rho^{(k)\circ} + \rho^{(k)'}, r^{(k)**} = r^{(k)*} + r^{(k)'} \end{aligned}\right) \tag{240}$$

$$\therefore \begin{aligned} \mathbf{u}_P^{(k)**} &= \mathbf{HP}_P^D\left[\mathbf{u}^{(k)**}\right] + \mathbf{HP}_P^C\left[\mathbf{u}^{(k)**}\right] - r_P^{(k)**}\mathbf{D}_P^{(k)}\nabla_P P^* \\ &= \mathbf{HP}_P^D\left[\mathbf{u}^{(k)*} + \mathbf{u}^{(k)'}\right] + \mathbf{HP}_P^C\left[\mathbf{u}^{(k)*} + \mathbf{u}^{(k)'}\right] - \left(r_P^{(k)*} + r_P^{(k)'}\right)\mathbf{D}_P^{(k)}\nabla_P\left(P^\circ + P'\right) \\ &= \mathbf{HP}_P^D\left[\mathbf{u}^{(k)*}\right] + \mathbf{HP}_P^D\left[\mathbf{u}^{(k)'}\right] + \mathbf{HP}_P^C\left[\mathbf{u}^{(k)*}\right] + \mathbf{HP}_P^C\left[\mathbf{u}^{(k)'}\right] \\ &\quad - r_P^{(k)*}\mathbf{D}_P^{(k)}\nabla_P P^\circ - r_P^{(k)*}\mathbf{D}_P^{(k)}\nabla_P P' - r_P^{(k)'}\mathbf{D}_P^{(k)}\nabla_P P^\circ - r_P^{(k)'}\mathbf{D}_P^{(k)}\nabla_P P' \end{aligned} \tag{241}$$

$$\therefore \begin{cases} \mathbf{u}_P^{(k)'} = \mathbf{HP}_P[\mathbf{u}^{(k)'}] + \mathbf{HP}_P^C[\mathbf{u}^{(k)*} - \mathbf{u}^{(k)\circ}] - r_P^{(k)*}\mathbf{D}_P^{(k)}\nabla_P P' - r_P^{(k)'}\mathbf{D}_P^{(k)}\nabla_P P^\circ - r_P^{(k)'}\mathbf{D}_P^{(k)}\nabla_P P' \\ \rho^{(k)'} = C_\rho^{(k)} P' \\ r_P^{(k)'} = -R_P^{(k)}\left(\dfrac{r_P^{(k)*}\Omega_P}{\delta t}\rho_P^{(k)'} + \Delta_P\left[\left(r^{(k)*}\rho^{(k)\circ}\mathbf{u}^{(k)'}\right).\mathbf{S} + r^{(k)*}U^{(k)*}\rho^{(k)'}\right]\right) \end{cases} \tag{242}$$

<u>Condition:</u>

$$\sum_k \left\{r_P^{(k)'}\right\} = RESG_P \tag{243}$$

$$\therefore \sum_k \left\{ -R_P^{(k)} \left(\begin{array}{l} \dfrac{r_P^{(k)*}\Omega_P C_\rho^{(k)}}{\delta t} P_P' + \Delta_P\left[r^{(k)*}U^{(k)*}C_\rho^{(k)}P'\right] + \\ \Delta_P\left[r^{(k)*}\rho^{(k)\circ}\begin{pmatrix} \mathbf{HP}[\mathbf{u}^{(k)'}] + \mathbf{HP}^C[\mathbf{u}^{(k)*} - \mathbf{u}^{(k)\circ}] \\ -r^{(k)*}\mathbf{D}^{(k)}\nabla P' - r^{(k)'}\mathbf{D}^{(k)}\nabla P' \end{pmatrix}.\mathbf{S} \right] \\ + \dfrac{\left(r_P^{(k)*}\rho_P^{(k)\circ}\right) - \left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega_P + \Delta_P\left[\left(r^{(k)*}\rho^{(k)\circ}U^{(k)*}\right)\right] \end{array} \right) \right\} = RESG_P \tag{244}$$

$$\therefore \begin{array}{l} \displaystyle\sum_k \left\{ R_P^{(k)}\left(\dfrac{r_P^{(k)*}\Omega_P C_\rho^{(k)}}{\delta t} P_P' + \Delta_P\left[r^{(k)*}U^{(k)*}C_\rho^{(k)}P'\right] - \Delta_P\left[r^{(k)*}\rho^{(k)\circ}\left(r^{(k)*}\mathbf{D}^{(k)}\nabla P'\right).\mathbf{S}\right] \right) \right\} = \\ \displaystyle -\sum_k \left\{ R_P^{(k)} \left(\begin{array}{l} \dfrac{\left(r_P^{(k)*}\rho_P^{(k)\circ}\right) - \left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega_P + \Delta_P\left[\left(r^{(k)*}\rho^{(k)\circ}U^{(k)*}\right)\right] + \\ \Delta_P\left[r^{(k)*}\rho^{(k)\circ}\left(\mathbf{HP}^C[\mathbf{u}^{(k)*} - \mathbf{u}^{(k)\circ}] + \mathbf{HP}[\mathbf{u}^{(k)'}] - r^{(k)'}\mathbf{D}^{(k)}\nabla P'\right).\mathbf{S}\right] \end{array} \right) \right\} - RESG_P \end{array} \tag{245}$$

<u>Approximation:</u>

Neglect: $\mathbf{HP}^C[\mathbf{u}^{(k)*} - \mathbf{u}^{(k)\circ}]$, $\mathbf{HP}[\mathbf{u}^{(k)'}]$, $r^{(k)'}\mathbf{D}^{(k)}\nabla P'$

$$\Rightarrow \mathbf{u}_P^{(k)'} = -r_P^{(k)*}\mathbf{D}_P^{(k)}\nabla_P P' \tag{246}$$

<u>Approximate Equation:</u>

$$\Rightarrow \begin{array}{l} \displaystyle\sum_k \left\{ R_P^{(k)}\left(\dfrac{r_P^{(k)*}\Omega_P C_\rho^{(k)}}{\delta t} P_P' + \Delta_P\left[r^{(k)*}U^{(k)*}C_\rho^{(k)}P'\right] - \Delta_P\left[r^{(k)*}\rho^{(k)\circ}\left(r^{(k)*}\mathbf{D}^{(k)}\nabla P'\right).\mathbf{S}\right] \right) \right\} = \\ \displaystyle -\sum_k \left\{ R_P^{(k)}\left(\dfrac{\left(r_P^{(k)*}\rho_P^{(k)\circ}\right) - \left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega_P + \Delta_P\left[\left(r^{(k)*}\rho^{(k)\circ}U^{(k)*}\right)\right] \right) \right\} - RESG_P \end{array} \tag{247}$$

A Global GCBA-SIMPLEST Iteration

- *Solve for* $r^{(k)}$*, treating* $H^D\left[r^{(k)}\right.$ *implicitly and* $H^C\left[r^{(k)}\right.$ *explicitly.*
- *Solve for* $\mathbf{u}^{(k)}$*, treating* $\mathbf{HP}^D\left[\mathbf{u}^{(k)}\right.$ *implicitly and* $\mathbf{HP}^C\left[\mathbf{u}^{(k)}\right.$ *explicitly.*
- *Calculate the* $\mathbf{D}^{(k)}$ *fields.*
- *Solve the pressure correction equation.*
- *Correct* $\mathbf{u}^{(k)}$*,* P*,* $r^{(k)}$ *and* $\rho^{(k)}$*.*
- *Solve implicitly the energy equations and update the density fields.*
- *Return to the first step and iterate until convergence.*

The GCBA following SIMPLER (GCBA-SIMPLER): Symbolic Form

First Predictor:

$$r_P^{(k)^*} = H_P[r^{(k)^*}] \tag{248}$$

Calculate the coefficients of the momentum equations.

First Corrector:

$$\left(\mathbf{u}^{(k)'}, P', \rho^{(k)'}, r^{(k)'}\right)\left(\begin{array}{l}\mathbf{u}^{(k)*} = \mathbf{u}^{(k)\circ} + \mathbf{u}^{(k)'}, P^* = P^\circ + P', \\ \rho^{(k)*} = \rho^{(k)\circ} + \rho^{(k)'}, r^{(k)**} = r^{(k)*} + r^{(k)'}\end{array}\right) \tag{249}$$

$$\therefore \mathbf{u}_P^{(k)^*} = \mathbf{HP}_P[\mathbf{u}^{(k)^*}] - r_P^{(k)^{**}}\mathbf{D}_P^{(k)}\nabla_P P^* = \mathbf{HP}_P[\mathbf{u}^{(k)\circ} + \mathbf{u}^{(k)'}] - \left(r_P^{(k)^*} + r_P^{(k)'}\right)\mathbf{D}_P^{(k)}\nabla_P\left(P^\circ + P'\right. \tag{250}$$

$$\mathbf{u}_P^{(k)\circ} = \mathbf{HP}_P\left[\mathbf{u}^{(k)\circ}\right] - r_P^{(k)^*}\mathbf{D}_P^{(k)}\nabla_P P^\circ \tag{251}$$

$$\therefore \left\{\begin{array}{l}\mathbf{u}_P^{(k)'} = \mathbf{HP}_P[\mathbf{u}^{(k)'}] - r_P^{(k)^*}\mathbf{D}_P^{(k)}\nabla_P P' - r_P^{(k)'}\mathbf{D}_P^{(k)}\nabla_P P^\circ - r_P^{(k)'}\mathbf{D}_P^{(k)}\nabla_P P' \\ \rho^{(k)'} = C_\rho^{(k)}P' \\ r_P^{(k)'} = -R_P^{(k)}\left(\dfrac{r_P^{(k)^*}\Omega_P}{\delta t}\rho_P^{(k)'} + \Delta_P\left[\left(r^{(k)^*}\rho^{(k)\circ}\mathbf{u}^{(k)'}\right)\mathbf{S} + r^{(k)^*}U^{(k)\circ}\rho^{(k)'}\right]\right)\end{array}\right. \tag{252}$$

Condition:

$$\sum_k\left\{r_P^{(k)'}\right\} = RESG_P \tag{253}$$

$$\therefore \sum_k\left\{-R_P^{(k)}\left(\begin{array}{l}\dfrac{r_P^{(k)^*}\Omega_P C_\rho^{(k)}}{\delta t}P_P' + \Delta_P\left[r^{(k)^*}U^{(k)\circ}C_\rho^{(k)}P'\right] + \\ \Delta_P\left[r^{(k)^*}\rho^{(k)\circ}\left(\mathbf{HP}[\mathbf{u}^{(k)'}] - r^{(k)^*}\mathbf{D}^{(k)}\nabla P' - r^{(k)'}\mathbf{D}^{(k)}\nabla P'\right)\mathbf{S}\right] \\ + \dfrac{\left(r_P^{(k)^*}\rho_P^{(k)\circ}\right) - \left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega_P + \Delta_P\left[\left(r^{(k)^*}\rho^{(k)\circ}U^{(k)\circ}\right)\right]\end{array}\right)\right\} = RESG_P \tag{254}$$

$$\begin{aligned}&\sum_k \left\{ R_P^{(k)} \left(\frac{r_P^{(k)*}\Omega_P C_\rho^{(k)}}{\delta t} P_P' + \Delta_P\left[r^{(k)*}U^{(k)\circ}C_\rho^{(k)}P'\right] - \Delta_P\left[r^{(k)*}\rho^{(k)\circ}\left(r^{(k)*}\mathbf{D}^{(k)}\nabla P'\right)\mathbf{S}\right]\right)\right\} = \\ \therefore\quad & -\sum_k \left\{ R_P^{(k)} \left(\begin{array}{l} \dfrac{\left(r_P^{(k)*}\rho_P^{(k)\circ}\right) - \left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega_P + \Delta_P\left[\left(r^{(k)*}\rho^{(k)\circ}U^{(k)\circ}\right)\right] + \\ \Delta_P\left[r^{(k)*}\rho^{(k)\circ}\left(\mathbf{HP}[\mathbf{u}^{(k)'}] - r^{(k)'}\mathbf{D}^{(k)}\nabla P'\right)\mathbf{S}\right] \end{array}\right)\right\} - RESG_P\end{aligned} \tag{255}$$

Approximation:

Neglect: $\mathbf{HP}[\mathbf{u}^{(k)'}],\ r^{(k)'}\mathbf{D}^{(k)}\nabla P'$

$$\Rightarrow \mathbf{u}_P^{(k)'} = -r_P^{(k)*}\mathbf{D}_P^{(k)}\nabla_P P' \tag{256}$$

Approximate Equation:

$$\begin{aligned}&\sum_k \left\{ R_P^{(k)} \left(\frac{r_P^{(k)*}\Omega_P C_\rho^{(k)}}{\delta t} P_P' + \Delta_P\left[r^{(k)*}U^{(k)\circ}C_\rho^{(k)}P'\right] - \Delta_P\left[r^{(k)*}\rho^{(k)\circ}\left(r^{(k)*}\mathbf{D}^{(k)}\nabla P'\right)\mathbf{S}\right]\right)\right\} = \\ \Rightarrow\quad & -\sum_k \left\{ R_P^{(k)} \left(\frac{\left(r_P^{(k)*}\rho_P^{(k)\circ}\right) - \left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega_P + \Delta_P\left[\left(r^{(k)*}\rho^{(k)\circ}U^{(k)\circ}\right)\right]\right)\right\} - RESG_P\end{aligned} \tag{257}$$

Apply correction to pressure, density, and volume fraction fields.

Second Predictor:

$$\mathbf{u}_P^{(k)*} = \mathbf{HP}_P[\mathbf{u}^{(k)*}] - r_P^{(k)**}\mathbf{D}_P^{(k)}\nabla_P P^* \tag{258}$$

Second Corrector:

$$\left(\mathbf{u}^{(k)''}, P'', \rho^{(k)''}, r^{(k)''}\right)\left(\begin{array}{l}\mathbf{u}^{(k)**} = \mathbf{u}^{(k)*} + \mathbf{u}^{(k)''}, P^{**} = P^* + P'', \\ \rho^{(k)**} = \rho^{(k)*} + \rho^{(k)''}, r^{(k)***} = r^{(k)**} + r^{(k)''}\end{array}\right) \tag{259}$$

$$\begin{aligned}\mathbf{u}_P^{(k)**} &= \mathbf{HP}_P\left[\mathbf{u}^{(k)**}\right] - r_P^{(k)***}\mathbf{D}_P^{(k)}\nabla_P P^{**} \\ \therefore\quad &= \mathbf{HP}_P\left[\mathbf{u}^{(k)*} + \mathbf{u}^{(k)''}\right] - \left(r_P^{(k)**} + r_P^{(k)''}\right)\mathbf{D}_P^{(k)}\nabla_P\left(P^* + P''\right)\end{aligned} \tag{260}$$

$$\therefore\left\{\begin{array}{l}\mathbf{u}_P^{(k)''} = \mathbf{HP}_P[\mathbf{u}^{(k)''}] - r_P^{(k)**}\mathbf{D}_P^{(k)}\nabla_P P'' - r_P^{(k)''}\mathbf{D}_P^{(k)}\nabla_P P^* - r_P^{(k)''}\mathbf{D}_P^{(k)}\nabla_P P'' \\ \rho^{(k)''} = C_\rho^{(k)}P'' \\ r_P^{(k)''} = -R_P^{(k)}\left(\dfrac{r_P^{(k)**}\Omega_P}{\delta t}\rho_P^{(k)''} + \Delta_P\left[\left(r^{(k)**}\rho^{(k)*}\mathbf{u}^{(k)''}\right)\mathbf{S} + r^{(k)**}U^{(k)*}\rho^{(k)''}\right]\right)\end{array}\right. \tag{261}$$

Condition:

$$\sum_k \left\{r_P^{(k)''}\right\} = RESG_P \tag{262}$$

$$\therefore \sum_k \left\{ -R_P^{(k)} \left(\begin{array}{l} \frac{r_P^{(k)**}\Omega_P C_\rho^{(k)}}{\delta t} P_P'' + \Delta_P\left[r^{(k)**}U^{(k)*}C_\rho^{(k)}P'\right] + \\ \Delta_P\left[r^{(k)**}\rho^{(k)*}\left(\mathbf{HP}[\mathbf{u}^{(k)''}] - r^{(k)**}\mathbf{D}^{(k)}\nabla_P P'' - r^{(k)''}\mathbf{D}^{(k)}\nabla P''\right)\mathbf{.S}\right] \\ + \frac{\left(r_P^{(k)**}\rho_P^{(k)*}\right) - \left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega_P + \Delta_P\left[\left(r^{(k)**}\rho^{(k)*}U^{(k)*}\right)\right] \end{array} \right) \right\} = RESG_P \tag{263}$$

$$\begin{array}{l} \sum_k \left\{ R_P^{(k)} \left(\frac{r_P^{(k)**}\Omega_P C_\rho^{(k)}}{\delta t} P_P'' + \Delta_P\left[r^{(k)**}U^{(k)*}C_\rho^{(k)}P''\right] - \Delta_P\left[r^{(k)**}\rho^{(k)*}\left(r^{(k)**}\mathbf{D}^{(k)}\nabla P''\right)\mathbf{.S}\right] \right) \right\} = \\ \therefore \quad -\sum_k \left\{ R_P^{(k)} \left(\begin{array}{l} \frac{\left(r_P^{(k)**}\rho_P^{(k)*}\right) - \left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega_P + \Delta_P\left[\left(r^{(k)**}\rho^{(k)*}U^{(k)*}\right)\right] + \\ \Delta_P\left[r^{(k)**}\rho^{(k)*}\left(\mathbf{HP}[\mathbf{u}^{(k)''}] - r^{(k)''}\mathbf{D}^{(k)}\nabla P''\right)\mathbf{.S}\right] \end{array} \right) \right\} - RESG_P \end{array} \tag{264}$$

<u>Approximation:</u>

Neglect: $\mathbf{HP}[\mathbf{u}^{(k)''}]$, $r^{(k)''}\mathbf{D}^{(k)}\nabla P''$

$$\Rightarrow \mathbf{u}_P^{(k)''} = -r_P^{(k)**}\mathbf{D}_P^{(k)}\nabla_P P'' \tag{265}$$

<u>Approximate Equation:</u>

$$\begin{array}{l} \sum_k \left\{ R_P^{(k)} \left(\frac{r_P^{(k)**}\Omega_P C_\rho^{(k)}}{\delta t} P_P'' + \Delta_P\left[r^{(k)**}U^{(k)*}C_\rho^{(k)}P''\right] - \Delta_P\left[r^{(k)**}\rho^{(k)*}\left(r^{(k)**}\mathbf{D}^{(k)}\nabla P''\right)\mathbf{.S}\right] \right) \right\} = \\ \Rightarrow \quad -\sum_k \left\{ R_P^{(k)} \left(\frac{\left(r_P^{(k)**}\rho_P^{(k)*}\right) - \left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega_P + \Delta_P\left[\left(r^{(k)**}\rho^{(k)*}U^{(k)*}\right)\right] \right) \right\} - RESG_P \end{array} \tag{266}$$

Apply correction to velocity fields.

A Global GCBA-SIMPLER Iteration

- Solve implicitly for $r^{(k)}$.
- Calculate the $\mathbf{D}^{(k)}$ fields.
- Solve the pressure correction equation and update the pressure, density, and volume fraction fields.
- Solve implicitly for $\mathbf{u}^{(k)}$ using the new pressure, density, and volume fraction fields
- Solve the pressure correction equation using the new velocity fields to obtain a new pressure correction field.
- Correct $\mathbf{u}^{(k)}$.
- Solve implicitly the energy equations and update the density fields.
- Return to the first step and iterate until convergence.

The GCBA following PISO (GCBA-PISO): Symbolic Form

First Predictor:

$$r_P^{(k)*} = H_P[r^{(k)*}] \tag{267}$$

$$\mathbf{u}_P^{(k)*} = \mathbf{HP}_P[\mathbf{u}^{(k)*}] - r_P^{(k)*}\mathbf{D}_P^{(k)}\nabla_P P^{\circ} \tag{268}$$

First Corrector:

$$\left(\mathbf{u}^{(k)'}, P', \rho^{(k)'}, r^{(k)'}\right)\begin{pmatrix} \mathbf{u}^{(k)**} = \mathbf{u}^{(k)*} + \mathbf{u}^{(k)'}, P^* = P^{\circ} + P', \\ \rho^{(k)*} = \rho^{(k)\circ} + \rho^{(k)'}, r^{(k)**} = r^{(k)*} + r^{(k)'} \end{pmatrix} \tag{269}$$

$$\therefore \mathbf{u}_P^{(k)**} = \mathbf{HP}_P[\mathbf{u}^{(k)**}] - r_P^{(k)**}\mathbf{D}_P^{(k)}\nabla_P P^* = \mathbf{HP}_P[\mathbf{u}^{(k)*} + \mathbf{u}^{(k)'}] - \left(r_P^{(k)*} + r_P^{(k)'}\right)\mathbf{D}_P^{(k)}\nabla_P\left(P^{\circ} + P'\right) \tag{270}$$

$$\therefore \begin{cases} \mathbf{u}_P^{(k)'} = \mathbf{HP}_P[\mathbf{u}^{(k)'}] - r_P^{(k)*}\mathbf{D}_P^{(k)}\nabla_P P' - r_P^{(k)'}\mathbf{D}_P^{(k)}\nabla_P P^{\circ} - r_P^{(k)'}\mathbf{D}_P^{(k)}\nabla_P P' \\ \rho^{(k)'} = C_\rho^{(k)} P' \\ r_P^{(k)'} = -R_P^{(k)}\left(\dfrac{r_P^{(k)*}\Omega_P}{\delta t}\rho_P^{(k)'} + \Delta_P\left[\left(r^{(k)*}\rho^{(k)\circ}\mathbf{u}^{(k)'}\right)\mathbf{S} + r^{(k)*}U^{(k)*}\rho^{(k)'}\right]\right) \end{cases} \tag{271}$$

Condition:

$$\sum_k \left\{r_P^{(k)'}\right\} = RESG_P \tag{272}$$

$$\therefore \sum_k \left\{ -R_P^{(k)} \begin{pmatrix} \dfrac{r_P^{(k)*}\Omega_P C_\rho^{(k)}}{\delta t} P_P' + \Delta_P\left[r^{(k)*}U^{(k)*}C_\rho^{(k)}P'\right] + \\ \Delta_P\left[r^{(k)*}\rho^{(k)\circ}\left(\mathbf{HP}[\mathbf{u}^{(k)'}] - r^{(k)*}\mathbf{D}^{(k)}\nabla P' - r^{(k)'}\mathbf{D}^{(k)}\nabla P'\right)\mathbf{S}\right] \\ + \dfrac{\left(r_P^{(k)*}\rho_P^{(k)\circ}\right) - \left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega_P + \Delta_P\left[\left(r^{(k)*}\rho^{(k)\circ}U^{(k)*}\right)\right] \end{pmatrix} \right\} = RESG_P \tag{273}$$

$$\therefore \begin{aligned}&\sum_k\left\{R_P^{(k)}\left(\frac{r_P^{(k)*}\Omega_P C_\rho^{(k)}}{\delta t}P_P' + \Delta_P\left[r^{(k)*}U^{(k)*}C_\rho^{(k)}P'\right] - \Delta_P\left[r^{(k)*}\rho^{(k)\circ}\left(r^{(k)*}\mathbf{D}^{(k)}\nabla P'\right)\mathbf{S}\right]\right)\right\} = \\ &-\sum_k\left\{R_P^{(k)}\left(\begin{aligned}&\frac{\left(r_P^{(k)*}\rho_P^{(k)\circ}\right)-\left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega_P + \Delta_P\left[\left(r^{(k)*}\rho^{(k)\circ}U^{(k)*}\right)\right] + \\ &\Delta_P\left[r^{(k)*}\rho^{(k)\circ}\left(\mathbf{HP}[\mathbf{u}^{(k)'}] - r^{(k)'}\mathbf{D}^{(k)}\nabla P'\right)\mathbf{S}\right]\end{aligned}\right)\right\} - RESG_P\end{aligned} \tag{274}$$

Approximation:

Neglect: $\mathbf{HP}[\mathbf{u}^{(k)'}],\ r^{(k)'}\mathbf{D}^{(k)}\nabla P'$

$$\Rightarrow \mathbf{u}_P^{(k)'} = -r_P^{(k)*}\mathbf{D}_P^{(k)}\nabla_P P' \tag{275}$$

Approximate Equation:

$$\Rightarrow \begin{aligned}&\sum_k\left\{R_P^{(k)}\left(\frac{r_P^{(k)*}\Omega_P C_\rho^{(k)}}{\delta t}P_P' + \Delta_P\left[r^{(k)*}U^{(k)*}C_\rho^{(k)}P'\right] - \Delta_P\left[r^{(k)*}\rho^{(k)\circ}\left(r^{(k)*}\mathbf{D}^{(k)}\nabla P'\right)\mathbf{S}\right]\right)\right\} = \\ &-\sum_k\left\{R_P^{(k)}\left(\frac{\left(r_P^{(k)*}\rho_P^{(k)\circ}\right)-\left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega_P + \Delta_P\left[\left(r^{(k)*}\rho^{(k)\circ}U^{(k)*}\right)\right]\right)\right\} - RESG_P\end{aligned} \tag{276}$$

Second Predictor:

$$r_P^{(k)***} = H_P^{**}[r^{(k)**}] \tag{277}$$

$$\mathbf{u}_P^{(k)***} = \mathbf{HP}_P^{**}[\mathbf{u}^{(k)**}] - r_P^{(k)***}\mathbf{D}_P^{(k)**}\nabla_P P^* \tag{278}$$

Second Corrector:

$$\left(\mathbf{u}^{(k)''}, P'', \rho^{(k)''}, r^{(k)''}\right)\left(\begin{aligned}&\mathbf{u}^{(k)****} = \mathbf{u}^{(k)***} + \mathbf{u}^{(k)''}, P^{**} = P^* + P'', \\ &\rho^{(k)**} = \rho^{(k)*} + \rho^{(k)''}, r^{(k)****} = r^{(k)***} + r^{(k)''}\end{aligned}\right) \tag{279}$$

$$\therefore \mathbf{u}_P^{(k)****} = \mathbf{HP}_P^{**}[\mathbf{u}^{(k)****}] - r_P^{(k)****}\mathbf{D}_P^{(k)**}\nabla_P\left(P^* + P''\right) \tag{280}$$

$$\mathbf{u}_P^{(k)***} = \mathbf{HP}_P^{**}[\mathbf{u}^{(k)**}] - r_P^{(k)***}\mathbf{D}_P^{(k)**}\nabla_P P^* \tag{281}$$

$$\therefore \begin{cases}\mathbf{u}_P^{(k)''} = \mathbf{HP}_P^{**}[\mathbf{u}^{(k)****} - \mathbf{u}^{(k)**}] - \left(r_P^{(k)***} + r_P^{(k)''}\right)\mathbf{D}_P^{(k)**}\nabla_P\left(P^* + P''\right) + r_P^{(k)**}\mathbf{D}_P^{(k)**}\nabla_P P^* \\ = \left(\mathbf{HP}_P^{**}[\mathbf{u}^{(k)***} - \mathbf{u}^{(k)**} + \mathbf{u}^{(k)''}] - r_P^{(k)''}\mathbf{D}_P^{(k)**}\nabla_P P'' - r_P^{(k)***}\mathbf{D}_P^{(k)**}\nabla_P P'' - r_P^{(k)''}\mathbf{D}_P^{(k)**}\nabla_P P^*\right. \\ \rho^{(k)''} = C_\rho^{(k)}P'' \\ r_P^{(k)''} = -R_P^{(k)**}\left(\dfrac{r_P^{(k)***}\Omega_P}{\delta t}\rho_P^{(k)''} + \Delta_P\left[\left(r^{(k)***}\rho^{(k)*}\mathbf{u}^{(k)''}\right)\mathbf{S} + r^{(k)***}U^{(k)***}\rho^{(k)''}\right]\right)\end{cases} \tag{282}$$

Condition:

$$\sum_k\left\{r_P^{(k)''}\right\} = 1 - \sum_k r_P^{(k)***} = RESG_P \tag{283}$$

$$\sum_k \left\{ -R_P^{(k)**} \left(\begin{array}{l} \dfrac{r_P^{(k)***}\Omega_P C_\rho^{(k)}}{\delta t} P_P'' + \Delta_P\left[r^{(k)***} U^{(k)***} C_\rho^{(k)} P'' \right] + \\ \Delta_P \left[r^{(k)***} \rho^{(k)*} \begin{pmatrix} \mathbf{HP}^{**}[\mathbf{u}^{(k)***} - \mathbf{u}^{(k)**} + \mathbf{u}^{(k)''}] \\ -r^{(k)''}\mathbf{D}^{(k)**}\nabla P'' - r^{(k)***}\mathbf{D}^{(k)**}\nabla P'' \end{pmatrix} .\mathbf{S} \right] \\ + \dfrac{\left(r_P^{(k)***}\rho_P^{(k)*}\right) - \left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t} \Omega_P + \Delta_P\left[\left(r^{(k)***}\rho^{(k)*} U^{(k)***} \right)\right] \end{array} \right) \right\} = RESG_P \tag{284}$$

$$\begin{array}{l} \sum_k \left\{ R_P^{(k)**} \left(\begin{array}{l} \dfrac{r_P^{(k)***}\Omega_P C_\rho^{(k)}}{\delta t} P_P'' + \Delta_P\left[r^{(k)***} U^{(k)***} C_\rho^{(k)} P'' \right] - \\ \Delta_P\left[r^{(k)***}\rho^{(k)*}\left(r^{(k)***}\mathbf{D}^{(k)**}\nabla P'' \right)\mathbf{S} \right] \end{array} \right) \right\} = \\ \therefore \\ - \sum_k \left\{ R_P^{(k)**} \left(\begin{array}{l} \dfrac{\left(r_P^{(k)***}\rho_P^{(k)*}\right) - \left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t} \Omega_P + \Delta_P\left[\left(r^{(k)***}\rho^{(k)*} U^{(k)***} \right)\right] + \\ \Delta_P \left[r^{(k)***} \rho^{(k)*} \begin{pmatrix} \mathbf{HP}^{**}[\mathbf{u}^{(k)***} - \mathbf{u}^{(k)**} + \mathbf{u}^{(k)''}] \\ -r^{(k)''}\mathbf{D}^{(k)**}\nabla P'' \end{pmatrix} .\mathbf{S} \right] \end{array} \right) \right\} - RESG_P \end{array} \tag{285}$$

Approximation:

Neglect: $\mathbf{HP}^{**}[\mathbf{u}^{(k)***} - \mathbf{u}^{(k)**} + \mathbf{u}^{(k)''}]$, $r^{(k)''}\mathbf{D}^{(k)**}\nabla P''$

$$\Rightarrow \mathbf{u}_P^{(k)''} = -r_P^{(k)\circ}\mathbf{D}_P^{(k)**}\nabla_P P'' \tag{286}$$

Approximate Equation:

$$\begin{array}{l} \sum_k \left\{ R_P^{(k)**} \left(\begin{array}{l} \dfrac{r_P^{(k)***}\Omega_P C_\rho^{(k)}}{\delta t} P_P'' + \Delta_P\left[r^{(k)***} U^{(k)***} C_\rho^{(k)} P'' \right] - \\ \Delta_P\left[r^{(k)***}\rho^{(k)*}\left(r^{(k)***}\mathbf{D}^{(k)**}\nabla P'' \right)\mathbf{S} \right] \end{array} \right) \right\} = \\ - \sum_k \left\{ R_P^{(k)**} \left(\dfrac{\left(r_P^{(k)***}\rho_P^{(k)*}\right) - \left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t} \Omega_P + \Delta_P\left[\left(r^{(k)***}\rho^{(k)*} U^{(k)***} \right)\right] \right) \right\} - RESG_P \end{array} \tag{287}$$

A Global GCBA-PISO Iteration

- *Solve implicitly for* $r^{(k)}$.
- *Solve implicitly for* $\mathbf{u}^{(k)}$ *using the old pressure and density fields.*
- *Calculate the* $\mathbf{D}^{(k)}$ *fields.*
- *Solve the pressure correction equation.*
- *Correct* $\mathbf{u}^{(k)}$, P, $r^{(k)}$, *and* $\rho^{(k)}$.
- *Solve implicitly the energy equation and update the density fields.*
- *Solve explicitly for* $r^{(k)}$.
- *Solve the momentum equations explicitly and calculate the* $\mathbf{D}^{(k)}$ *fields.*
- *Solve the pressure correction equation.*
- *Correct* $\mathbf{u}^{(k)}$, P, $r^{(k)}$, *and* $\rho^{(k)}$.
- *Return to step one and iterate until convergence*

The GCBA following SIMPLEX (GCBA-SIMPLEX): Symbolic Form

Predictor:

$$r_P^{(k)*} = H_P[r^{(k)*}] \tag{288}$$

$$\mathbf{u}_P^{(k)*} = \mathbf{HP}_P[\mathbf{u}^{(k)*}] - r_P^{(k)*}\mathbf{D}_P^{(k)}\nabla_P P^{\circ} \tag{289}$$

Corrector:

$$\left(\mathbf{u}^{(k)'}, P', \rho^{(k)'}, r^{(k)'}\right)\begin{pmatrix} \mathbf{u}^{(k)**} = \mathbf{u}^{(k)*} + \mathbf{u}^{(k)'}, P^* = P^{\circ} + P', \\ \rho^{(k)*} = \rho^{(k)\circ} + \rho^{(k)'}, r^{(k)**} = r^{(k)*} + r^{(k)'} \end{pmatrix} \tag{290}$$

$$\therefore \mathbf{u}_P^{(k)**} = \mathbf{HP}_P[\mathbf{u}^{(k)**}] - r_P^{(k)**}\mathbf{D}_P^{(k)}\nabla_P P^* = \mathbf{HP}_P[\mathbf{u}^{(k)*} + \mathbf{u}^{(k)'}] - \left(r_P^{(k)*} + r_P^{(k)'}\right)\mathbf{D}_P^{(k)}\nabla_P\left(P^{\circ} + P'\right) \tag{291}$$

$$\therefore \begin{cases} \mathbf{u}_P^{(k)'} = \mathbf{HP}_P[\mathbf{u}^{(k)'}] - r_P^{(k)*}\mathbf{D}_P^{(k)}\nabla_P P' - r_P^{(k)'}\mathbf{D}_P^{(k)}\nabla_P P^{\circ} - r_P^{(k)'}\mathbf{D}_P^{(k)}\nabla_P P' \\ \rho^{(k)'} = C_\rho^{(k)} P' \\ r_P^{(k)'} = -R_P^{(k)}\left(\dfrac{r_P^{(k)*}\Omega_P}{\delta t}\rho_P^{(k)'} + \Delta_P\left[\left(r^{(k)*}\rho^{(k)\circ}\mathbf{u}^{(k)'}\right)\mathbf{S} + r^{(k)*}U^{(k)*}\rho^{(k)'}\right]\right) \end{cases} \tag{292}$$

Condition:

$$\sum_k \left\{r_P^{(k)'}\right\} = RESG_P \tag{293}$$

$$\therefore \sum_k \left\{ -R_P^{(k)} \begin{pmatrix} \dfrac{r_P^{(k)*}\Omega_P C_\rho^{(k)}}{\delta t} P_P' + \Delta_P\left[r^{(k)*}U^{(k)*}C_\rho^{(k)}P'\right] + \\ \Delta_P\left[r^{(k)*}\rho^{(k)\circ}\left(\mathbf{HP}_P[\mathbf{u}^{(k)'}] - r_P^{(k)*}\mathbf{D}_P^{(k)}\nabla_P P' - r_P^{(k)'}\mathbf{D}_P^{(k)}\nabla_P P'\right)\mathbf{S}\right] \\ + \dfrac{\left(r_P^{(k)*}\rho_P^{(k)\circ}\right) - \left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega_P + \Delta_P\left[\left(r^{(k)*}\rho^{(k)\circ}U^{(k)*}\right)\right] \end{pmatrix} \right\} = RESG_P \tag{294}$$

Approximation:

Neglect: $r_P^{(k)'}\mathbf{D}_P^{(k)}\nabla_P P'$, and let

$$\mathbf{u}_P^{(k)'} = \mathbf{HP}_P[\mathbf{u}^{(k)'}] - r_P^{(k)*}\mathbf{D}_P^{(k)}\nabla_P P' = -r_P^{(k)*}\mathbf{D}_P^{(k)SX}\nabla_P P' \tag{295}$$

$$\Rightarrow -r_P^{(k)*}\mathbf{D}_P^{(k)SX}\nabla_P P' = \mathbf{HP}_P[-r^{(k)*}\mathbf{D}^{(k)SX}\nabla P'] - r_P^{(k)*}\mathbf{D}_P^{(k)}\nabla_P P' \tag{296}$$

Assume that the pressure difference local to the velocity is representative of all pressure differences i.e. $\mathbf{HP}_P[-r^{(k)*}\mathbf{D}^{(k)SX}\nabla P'] = -(\nabla_P P')\mathbf{HP}_P[r^{(k)*}\mathbf{D}^{(k)SX}]$, thus:

$$-r_P^{(k)*}\mathbf{D}_P^{(k)SX}(\nabla_P P') = -(\nabla_P P')\mathbf{HP}_P[r^{(k)*}\mathbf{D}^{(k)SX}] - r_P^{(k)*}\mathbf{D}_P^{(k)}(\nabla_P P') \tag{297}$$

$$\Rightarrow r_P^{(k)*}\mathbf{D}_P^{(k)SX} = \mathbf{HP}_P[r^{(k)*}\mathbf{D}^{(k)SX}] + r_P^{(k)*}\mathbf{D}_P^{(k)} \tag{298}$$

Approximate Equation:

$$\Rightarrow \begin{array}{l} \sum_k \left\{ R_P^{(k)}\left(\dfrac{r_P^{(k)*}\Omega_P C_\rho^{(k)}}{\delta t} P_P' + \Delta_P\left[r^{(k)*}U^{(k)*}C_\rho^{(k)}P'\right] - \Delta_P\left[r^{(k)*}\rho^{(k)\circ}\left(r^{(k)*}\mathbf{D}^{(k)SX}\nabla P'\right).\mathbf{S}\right]\right)\right\} = \\ -\sum_k \left\{ R_P^{(k)}\left(\dfrac{\left(r_P^{(k)*}\rho_P^{(k)\circ}\right) - \left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega_P + \Delta_P\left[\left(r^{(k)*}\rho^{(k)\circ}U^{(k)*}\right)\right]\right)\right\} - RESG_P \end{array} \tag{299}$$

A Global GCBA-SIMPLEX Iteration

- *Solve implicitly for* $r^{(k)}$.
- *Solve implicitly for* $\mathbf{u}^{(k)}$, *using the old pressure and density fields.*
- *Calculate the* $\mathbf{D}^{(k)}$ *fields.*
- *Solve implicitly for the* $\mathbf{D}^{(k)SX}$ *fields.*
- *Solve the pressure correction equation using these* $\mathbf{D}^{(k)SX}$ *fields.*
- *Correct* $\mathbf{u}^{(k)}$, P, $r^{(k)}$, *and* $\rho^{(k)}$.
- *Solve implicitly the energy equations and update the density fields.*
- *Return to the first step and iterate until convergence.*

The GCBA following SIMPLEM (GCBA-SIMPLEM): Symbolic Form

First Predictor:

$$r_P^{(k)*} = H_P[r^{(k)*}] \tag{300}$$

Calculate the coefficients of the momentum equations.

First Corrector:

$$\left(\mathbf{u}^{(k)'}, P', \rho^{(k)'}, r^{(k)'}\right)\left(\begin{array}{l}\mathbf{u}^{(k)*} = \mathbf{u}^{(k)\circ} + \mathbf{u}^{(k)'}, P^* = P^\circ + P', \\ \rho^{(k)*} = \rho^{(k)\circ} + \rho^{(k)'}, r^{(k)**} = r^{(k)*} + r^{(k)'}\end{array}\right) \tag{301}$$

$$\therefore \mathbf{u}_P^{(k)*} = \mathbf{HP}_P[\mathbf{u}^{(k)*}] - r_P^{(k)**}\mathbf{D}_P^{(k)}\nabla_P P^* = \mathbf{HP}_P[\mathbf{u}^{(k)\circ} + \mathbf{u}^{(k)'}] - \left(r_P^{(k)*} + r_P^{(k)'}\right)\mathbf{D}_P^{(k)}\nabla_P\left(P^\circ + P'\right) \quad (302)$$

$$\mathbf{u}_P^{(k)\circ} = \mathbf{HP}_P\left[\mathbf{u}^{(k)\circ}\right] - r_P^{(k)*}\mathbf{D}_P^{(k)}\nabla_P P^\circ \quad (303)$$

$$\therefore \begin{cases} \mathbf{u}_P^{(k)'} = \mathbf{HP}_P[\mathbf{u}^{(k)'}] - r_P^{(k)*}\mathbf{D}_P^{(k)}\nabla_P P' - r_P^{(k)'}\mathbf{D}_P^{(k)}\nabla_P P^\circ - r_P^{(k)'}\mathbf{D}_P^{(k)}\nabla_P P' \\ \rho^{(k)'} = C_\rho^{(k)} P' \\ r_P^{(k)'} = -R_P^{(k)}\left(\dfrac{r_P^{(k)*}\Omega_P}{\delta t}\rho_P^{(k)'} + \Delta_P\left[\left(r^{(k)*}\rho^{(k)\circ}\mathbf{u}^{(k)'}\right)\mathbf{S} + r^{(k)*}U^{(k)\circ}\rho^{(k)'}\right]\right) \end{cases} \quad (304)$$

Condition:

$$\sum_k \left\{ r_P^{(k)'} \right\} = RESG_P \quad (305)$$

$$\therefore \sum_k \left\{ -R_P^{(k)} \left(\begin{array}{l} \dfrac{r_P^{(k)*}\Omega_P C_\rho^{(k)}}{\delta t} P_P' + \Delta_P\left[r^{(k)*}U^{(k)\circ}C_\rho^{(k)}P'\right] + \\ \Delta_P\left[r^{(k)*}\rho^{(k)\circ}\left(\mathbf{HP}[\mathbf{u}^{(k)'}] - r^{(k)*}\mathbf{D}^{(k)}\nabla P' - r^{(k)'}\mathbf{D}^{(k)}\nabla P'\right)\mathbf{S}\right] \\ + \dfrac{\left(r_P^{(k)*}\rho_P^{(k)\circ}\right) - \left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega_P + \Delta_P\left[\left(r^{(k)*}\rho^{(k)\circ}U^{(k)\circ}\right)\right] \end{array} \right) \right\} = RESG_P \quad (306)$$

$$\therefore \begin{array}{l} \displaystyle\sum_k \left\{ R_P^{(k)} \left(\frac{r_P^{(k)*}\Omega_P C_\rho^{(k)}}{\delta t} P_P' + \Delta_P\left[r^{(k)*}U^{(k)\circ}C_\rho^{(k)}P'\right] - \Delta_P\left[r^{(k)*}\rho^{(k)\circ}\left(r^{(k)*}\mathbf{D}^{(k)}\nabla P'\right)\mathbf{S}\right] \right) \right\} = \\ \displaystyle -\sum_k \left\{ R_P^{(k)} \left(\begin{array}{l} \dfrac{\left(r_P^{(k)*}\rho_P^{(k)\circ}\right) - \left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega_P + \Delta_P\left[\left(r^{(k)*}\rho^{(k)\circ}U^{(k)\circ}\right)\right] + \\ \Delta_P\left[r^{(k)*}\rho^{(k)\circ}\left(\mathbf{HP}[\mathbf{u}^{(k)'}] - r^{(k)'}\mathbf{D}^{(k)}\nabla P'\right)\mathbf{S}\right] \end{array} \right) \right\} - RESG_P \end{array} \quad (307)$$

Approximation:

Neglect: $\mathbf{HP}[\mathbf{u}^{(k)'}]$, $r^{(k)'}\mathbf{D}^{(k)}\nabla P'$

$$\Rightarrow \mathbf{u}_P^{(k)'} = -r_P^{(k)*}\mathbf{D}_P^{(k)}\nabla_P P' \quad (308)$$

Approximate Equation:

$$\Rightarrow \begin{array}{l} \displaystyle\sum_k \left\{ R_P^{(k)} \left(\frac{r_P^{(k)*}\Omega_P C_\rho^{(k)}}{\delta t} P_P' + \Delta_P\left[r^{(k)*}U^{(k)\circ}C_\rho^{(k)}P'\right] - \Delta_P\left[r^{(k)*}\rho^{(k)\circ}\left(r^{(k)*}\mathbf{D}^{(k)}\nabla P'\right)\mathbf{S}\right] \right) \right\} = \\ \displaystyle -\sum_k \left\{ R_P^{(k)} \left(\frac{\left(r_P^{(k)*}\rho_P^{(k)\circ}\right) - \left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega_P + \Delta_P\left[\left(r^{(k)*}\rho^{(k)\circ}U^{(k)\circ}\right)\right] \right) \right\} - RESG_P \end{array} \quad (309)$$

Second Predictor:

$$\mathbf{u}_P^{(k)***} = \mathbf{HP}_P[\mathbf{u}^{(k)**}] - r_P^{(k)**}\mathbf{D}_P^{(k)*}\nabla_P P^* \tag{310}$$

Second Corrector:

No corrector stage.

A Global MCBA-SIMPLEM Iteration

- *Solve implicitly for* $r^{(k)}$.
- *Calculate the* $\mathbf{D}^{(k)}$ *fields based on values from the previous iteration.*
- *Solve the pressure correction equation.*
- *Correct* $\mathbf{u}^{(k)}$, P, $r^{(k)}$, *and* $\rho^{(k)}$.
- *Calculate new* $\mathbf{HP}^{(k)}$ *and* $\mathbf{D}^{(k)}$ *fields.*
- *Solve implicitly for* $\mathbf{u}^{(k)}$ *using the new fields.*
- *Solve implicitly the energy equations and update the density fields.*
- *Return to the first step and iterate until convergence.*

As for the MCBA, most of the GCBA algorithms have not been extended and tested in multi-fluid flow situations. As such, the only way to discuss their merit is by assuming that they behave as in single-fluid flows. Since such a discussion was given earlier, it will not be repeated here.

The Expanded Form of the Pressure-Correction Equation

The expanded form of the pressure correction equation, applicable to all algorithms, will be presented. For that purpose, let $r^{(k)}$, $U^{(k)}$ and $\rho^{(k)}$ denote values from the previous iteration or from a previous corrector step, then, the pressure correction equation becomes

$$\begin{aligned}
&\sum_k \left\{ R_P^{(k)} \left(\frac{r_P^{(k)} C_\rho^{(k)} \Omega_P}{\delta t} P_P' + \Delta_P\left[r^{(k)} U^{(k)} C_\rho^{(k)} P'\right] - \Delta_P\left[r^{(k)}\rho^{(k)}\left(r^{(k)}\mathbf{D}^{(k)}\nabla(P')\right)\mathbf{S}\right]\right)\right\} = \\
&\quad -\sum_{k=1}^{nphase}\left\{ R_P^{(k)}\left(\frac{\left(r_P^{(k)}\rho_P^{(k)}\right) - \left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega_P + \Delta_P\left[\left(r^{(k)}\rho^{(k)}U^{(k)}\right)\right]\right)\right\} - RESG_P
\end{aligned} \tag{311}$$

In this form, the resemblance between this equation and the MCBA pressure correction equation is obvious. The discretization of the above equation yields

$$A_P^{P'}P_P' = A_E^{P'}P_E' + A_W^{P'}P_W' + A_N^{P'}P_N' + A_S^{P'}P_S' + A_T^{P'}P_T' + A_B^{P'}P_B' + B_P^{P'} \tag{312}$$

where

$$
\begin{aligned}
A_E^{P'} &= \sum_k R_P^{(k)} \left[\Gamma_e^{(k)} + \left(r^{(k)} C_\rho^{(k)} \right)_e \left\| -U_e^{(k)}, 0 \right\| \right] \\
A_W^{P'} &= \sum_k R_P^{(k)} \left[\Gamma_w^{(k)} + \left(r^{(k)} C_\rho^{(k)} \right)_w \left\| -U_w^{(k)}, 0 \right\| \right] \\
A_N^{P'} &= \sum_k R_P^{(k)} \left[\Gamma_n^{(k)} + \left(r^{(k)} C_\rho^{(k)} \right)_n \left\| -U_n^{(k)}, 0 \right\| \right] \\
A_S^{P'} &= \sum_k R_P^{(k)} \left[\Gamma_s^{(k)} + \left(r^{(k)} C_\rho^{(k)} \right)_s \left\| -U_s^{(k)}, 0 \right\| \right] \\
A_T^{P'} &= \sum_k R_P^{(k)} \left[\Gamma_t^{(k)} + \left(r^{(k)} C_\rho^{(k)} \right)_t \left\| -U_t^{(k)}, 0 \right\| \right] \\
A_B^{P'} &= \sum_k R_P^{(k)} \left[\Gamma_b^{(k)} + \left(r^{(k)} C_\rho^{(k)} \right)_b \left\| -U_b^{(k)}, 0 \right\| \right] \\
A_P^{P'} &= A_E^{P'} + A_W^{P'} + A_N^{P'} + A_S^{P'} + A_T^{P'} + A_B^{P'} + \\
&\sum_k R_P^{(k)} \left(\begin{array}{l} \dfrac{r_P^{(k)} K_P^{(k)} \Omega_P}{\delta t} + \left(r^{(k)} C_\rho^{(k)} \right)_e U_s^{(k)} + \left(r^{(k)} C_\rho^{(k)} \right)_w U_w^{(k)} + \left(r^{(k)} C_\rho^{(k)} \right)_n U_n^{(k)} \\ \qquad + \left(r^{(k)} C_\rho^{(k)} \right)_s U_s^{(k)} + \left(r^{(k)} C_\rho^{(k)} \right)_t U_t^{(k)} + \left(r^{(k)} C_\rho^{(k)} \right)_b U_b^{(k)} \end{array} \right) \\
B_P^{P'} &= -\sum_k R_P^{(k)} \left\{ \begin{array}{l} \dfrac{\left(r_P^{(k)} \rho_P^{(k)} - r_P^{(k)old} \rho_P^{(k)old} \right)}{\delta t} \Omega_P \\ + \left(\begin{array}{l} r_e^k \rho_e^{(k)} U_s^{(k)} + r_w^k \rho_w^{(k)} U_w^{(k)} + r_n^k \rho_n^{(k)} U_n^{(k)} \\ + r_s^k \rho_s^{(k)} U_s^{(k)} + r_t^k \rho_t^{(k)} U_t^{(k)} + r_b^k \rho_b^{(k)} U_b^{(k)} \end{array} \right) \end{array} \right\} - RESG_P
\end{aligned}
\tag{313}
$$

Following the calculation of the pressure correction field, $\mathbf{u}_P^{(k)'}$, $\rho_P^{(k)'}$, and $r_P^{(k)'}$ are obtained using the following equations

$$
\begin{aligned}
&\mathbf{u}_P^{(k)'} = -r^{(k)} \mathbf{D}_P^{(k)} \nabla_P (P') \\
&\therefore \rho^{(k)'} = C_\rho^{(k)} P' \\
&r_P^{(k)'} = -R_P^{(k)} \left(\frac{r_P^{(k)} \Omega_P}{\delta t} \rho_P^{(k)'} + \Delta_P \left[\left(r^{(k)} \rho^{(k)} \mathbf{u}^{(k)'} \right) \mathbf{S} + r^{(k)} U^{(k)} \rho^{(k)'} \right] \right)
\end{aligned}
\tag{314}
$$

Improvement #6

The pressure correction equation was derived in the previous section by neglecting the $H_P\left[r^{(k)'}\right.$ term. Following the SIMPLEC methodology, a better approximation can be achieved by adding and subtracting $H_P[1]r_P^{(k)'}$ from the left-hand side of equation (199), which results in neglecting a smaller term $\left(H_P\left[r^{(k)'} - r_P^{(k)'}\right]\right.$. With this approximation, the pressure correction equation becomes:

$$\left(1-H_P[1]\right)r_P^{(k)'}=$$
$$-R_P^{(k)}\left(\begin{array}{l}\dfrac{r_P^{(k)*}\Omega_P}{\delta t}\rho_P^{(k)'}+\Delta_P\left[\begin{array}{l}r^{(k)*}\rho^{(k)\circ}\left(\mathbf{HP}[\mathbf{u}^{(k)'}]-r^{(k)*}\mathbf{D}^{(k)}\nabla P'-r^{(k)'}\mathbf{D}^{(k)}\nabla P'\right)\mathbf{S}\\+r^{(k)*}U^{(k)*}\rho^{(k)'}\end{array}\right]\\+\dfrac{\left(r_P^{(k)*}\rho_P^{(k)\circ}\right)-\left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega_P+\Delta_P\left[\left(r^{(k)*}\rho^{(k)\circ}U^{(k)*}\right)\right]+\dot{M}_P^{(k)}r_P^{(k)*}\Omega_P\end{array}\right) \tag{315}$$

or

$$r_P^{(k)'}=-\tilde{R}_P^{(k)}\left(\begin{array}{l}\dfrac{r_P^{(k)*}\Omega_P}{\delta t}\rho_P^{(k)'}+\Delta_P\left[\begin{array}{l}r^{(k)*}\rho^{(k)\circ}\left(\mathbf{HP}[\mathbf{u}^{(k)'}]-r^{(k)*}\mathbf{D}^{(k)}\nabla P'-r^{(k)'}\mathbf{D}^{(k)}\nabla P'\right)\mathbf{S}\\+r^{(k)*}U^{(k)*}\rho^{(k)'}\end{array}\right]\\+\dfrac{\left(r_P^{(k)*}\rho_P^{(k)\circ}\right)-\left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega_P+\Delta_P\left[\left(r^{(k)*}\rho^{(k)\circ}U^{(k)*}\right)\right]+\dot{M}_P^{(k)}r_P^{(k)*}\Omega_P\end{array}\right) \tag{316}$$

Where

$$\tilde{R}_P^{(k)}=\frac{R_P^{(k)}}{\left(1-H_P[1]\right)} \tag{317}$$

The pressure correction equation is obtained by substituting Eq. (316) into the geometric conservation equation. The expanded form of the resulting pressure correction equation is easily obtained from Eqs. (312) and (313) by simply substituting $\tilde{R}_P^{(k)}$ for $R_P^{(k)}$. Furthermore, depending on the approximation made to the $\mathbf{HP}\left[\mathbf{u}^{(k)'}\right]$ term, a new family of GCBA, similar to the one detailed above, can be obtained.

Part III – Comparing GCBA and MCBA Formulations

Scaling GCBA to Single-Fluid Flow Simulations

Since MCBA algorithms are derived through direct extension of the single fluid algorithms, they can scale down automatically to handle one-fluid simulations. On the other hand, because GCBA algorithms are based on geometric conservation (as opposed to mass conservation) their scaling to one-fluid simulations is not obvious. Such a property is useful in the sense that coding for single and multi-fluid models would follow the same structure.

To show how multi-fluid GCBA algorithms scale down to single-fluid flow simulations, equation (201) is reproduced as follows:

$$r_P^{(k)'} = -R_P^{(k)}\left(\begin{array}{l} \dfrac{r_P^{(k)*}\Omega_P}{\delta t}\rho_P^{(k)'} + \Delta_P\left[r^{(k)*}\rho^{(k)\circ}\left(\begin{array}{l}\mathbf{HP}[\mathbf{u}^{(k)'}] - r^{(k)*}\mathbf{D}^{(k)}\nabla P' \\ -r^{(k)'}\mathbf{D}^{(k)}\nabla P'\end{array}\right).\mathbf{S} + r^{(k)*}U^{(k)*}\rho^{(k)'}\right] \\ +\dfrac{\left(r_P^{(k)*}\rho_P^{(k)\circ}\right)-\left(r_P^{(k)}\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega_P + \Delta_P\left[\left(r^{(k)*}\rho^{(k)\circ}U^{(k)*}\right)\right] - \dot{M}_P^{(k)}r_P^{(k)*}\Omega_P \end{array}\right) \tag{318}$$

For the case of one fluid flow $r_P^{(k)}$=1=Cte, thus $r_P^{(k)'}$=0, removing $r_P^{(k)'}$ from equation (318) and setting $r_P^{(k)*}$ to 1 yields:

$$0 = -R_P^{(k)}\left(\begin{array}{l} \dfrac{\Omega_P}{\delta t}\rho_P^{(k)'} + \Delta_P\left[\rho^{(k)\circ}\left(\mathbf{HP}[\mathbf{u}^{(k)'}] - r^{(k)*}\mathbf{D}^{(k)}\nabla P' - r^{(k)'}\mathbf{D}^{(k)}\nabla P'\right)\mathbf{S} + U^{(k)*}\rho^{(k)'}\right] \\ +\dfrac{\left(\rho_P^{(k)\circ}\right)-\left(\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega_P + \Delta_P\left[\left(\rho^{(k)\circ}U^{(k)*}\right)\right] - \dot{M}_P^{(k)}\Omega_P \end{array}\right) \tag{319}$$

Removing $R_P^{(k)}$ and rearraging, the pressure correction equation transforms to

$$\begin{array}{l} \dfrac{\Omega_P}{\delta t}\rho_P^{(k)'} + \Delta_P\left[\rho^{(k)\circ}\left(\mathbf{HP}[\mathbf{u}^{(k)'}] - r^{(k)*}\mathbf{D}^{(k)}\nabla P' - r^{(k)'}\mathbf{D}^{(k)}\nabla P'\right)\mathbf{S} + U^{(k)*}\rho^{(k)'}\right. \\ \qquad = -\dfrac{\left(\rho_P^{(k)\circ}\right)-\left(\rho_P^{(k)}\right)^{Old}}{\delta t}\Omega_P - \Delta_P\left[\left(\rho^{(k)\circ}U^{(k)*}\right)\right] + \dot{M}_P^{(k)}\Omega_P \end{array} \tag{320}$$

The expansion of the above equation is straightforward and leads to the pressure-correction equation of a single-fluid flow.

Relating GCBA and MCBA

While the derivations of GCBA and MCBA are based on different paradigms, it is shown in this section that the GCBA formulation leads to a weighted pressure correction equation that has close similarity with the MCBA formulation to which improvement #2 has been applied.

The respective GCBA and MCBA (with improvement #2) pressure correction equations, as derived earlier, are as follows:

$$\sum_k \left\{ R_P^{(k)} \left(\frac{r_P^{(k)} C_\rho^{(k)} \Omega_P}{\delta t} P_P' + \Delta_P \left[r^{(k)} U^{(k)} C_\rho^{(k)} P' \right] - \Delta_P \left[r^{(k)} \rho^{(k)} \left(r^{(k)} \mathbf{D}^{(k)} \nabla (P') \right) \mathbf{S} \right] \right) \right\} = \\ - \sum_{k=1}^{nphase} \left\{ R_P^{(k)} \left(\frac{\left(r_P^{(k)} \rho_P^{(k)} \right) - \left(r_P^{(k)} \rho_P^{(k)} \right)^{Old}}{\delta t} \Omega_P + \Delta_P \left[\left(r^{(k)} \rho^{(k)} U^{(k)} \right) \right] \right) \right\} - RESG_P \tag{321}$$

$$\sum_k \left\{ \frac{\Omega}{\delta t \underline{\rho}^{(k)}} r_P^{(k)\circ} C_\rho^{(k)} P_P' + \Delta_P \left[\frac{r^{(k)\circ} C_\rho^{(k)} U^{(k)} P'}{\underline{\rho}^{(k)}} \right] - \Delta_P \left[r^{(k)\circ} \frac{\rho^{(k)}}{\underline{\rho}^{(k)}} \left(r^{(k)\circ} \mathbf{D}^{(k)} \nabla P' \right) \mathbf{S} \right] \right\} \\ = - \sum_k \left\{ \frac{r_P^{(k)\circ} \left(\rho_P^{(k)} / \underline{\rho}^{(k)} \right) - \left[r_P^{(k)} \left(\rho_P^{(k)} / \underline{\rho}^{(k)} \right) \right]^{Old}}{\delta t} \Omega + \Delta_P \left[r^{(k)\circ} \frac{\rho^{(k)}}{\underline{\rho}^{(k)}} U^{(k)} \right] \right\} \tag{322}$$

Upong comparing the two equations it is clear that $R_P^{(k)}$ plays the role of the reference density, i.e. a weighing factor for the respective phasic continuity equation. As such, the GCBA pressure correction equation is very similar to a weighted MCBA pressure correction equation. The weighing procedure (normalization) is done automatically based on the local strength of the inflow to the control volume (since $R_P^{(k)} = 1 / A_P^{(k)}$ of the volume fraction equation) thus, as $r_P^{(1)} >> r_P^{(2)}$ one gets $R_P^{(1)} << R_P^{(2)}$. This treatment yields a more robust behaviour since large fluid density differences will not mean that conservation for the lighter fluid is lost due to numerical errors.

The above equations also differ slightly in the source term, where an additional entry ($RESG_P$) is included in equation (321) to account for residuals of the geometric conservation equation. Hence, the family of MCBA algorithms can be viewed as a subset of the GCBA family, and could be recovered by setting the volume fraction corrections to zero ($r_P^{(k)'} = 0$). As such, codes based on the MCBA can easily cater for GCBA and vice versa.

The advantage of the GCBA over the MCBA is in its attempt to correct both the velocity and volume fraction fields. However there are still limitations pertaining to the calculation of the volume fraction field, namely that the calculation is done without accounting for the influence of the volumes fractions of the different fluids on each other. Thus while the

pressure change will tend to balance the volume fractions, it will not do so in an optimal way. In addition, all volume fractions will change in the same direction in response to pressure adjustment, which is not always logical.

Closing Remarks

The segregated class of single-fluid flow algorithms was extended to predict multi-fluid flow at all speeds. The formulation was done using a unified, compact, and easy to understand notation. Depending on the constraint equation used to derive the pressure or pressure correction equation, the extended algorithms were shown to fall under two categories that were denoted by the Mass Conservation Based Algorithms (MCBA) and the Geometric Conservation Based Algorithms (GCBA). The differences and similarities between the two categories were explained. In addition, several techniques developed to promote and accelerate the convergence of these algorithms were also discussed.

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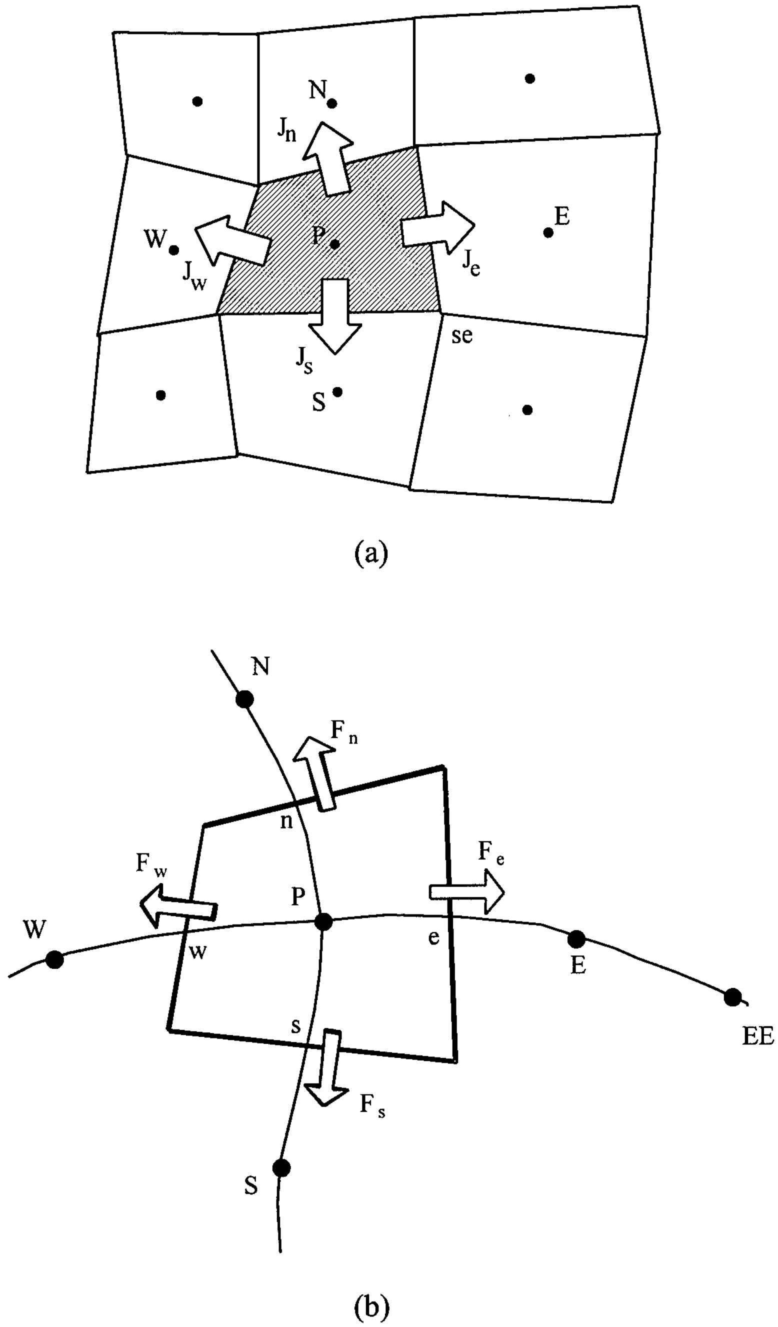

Fig. 1 Control Volume.

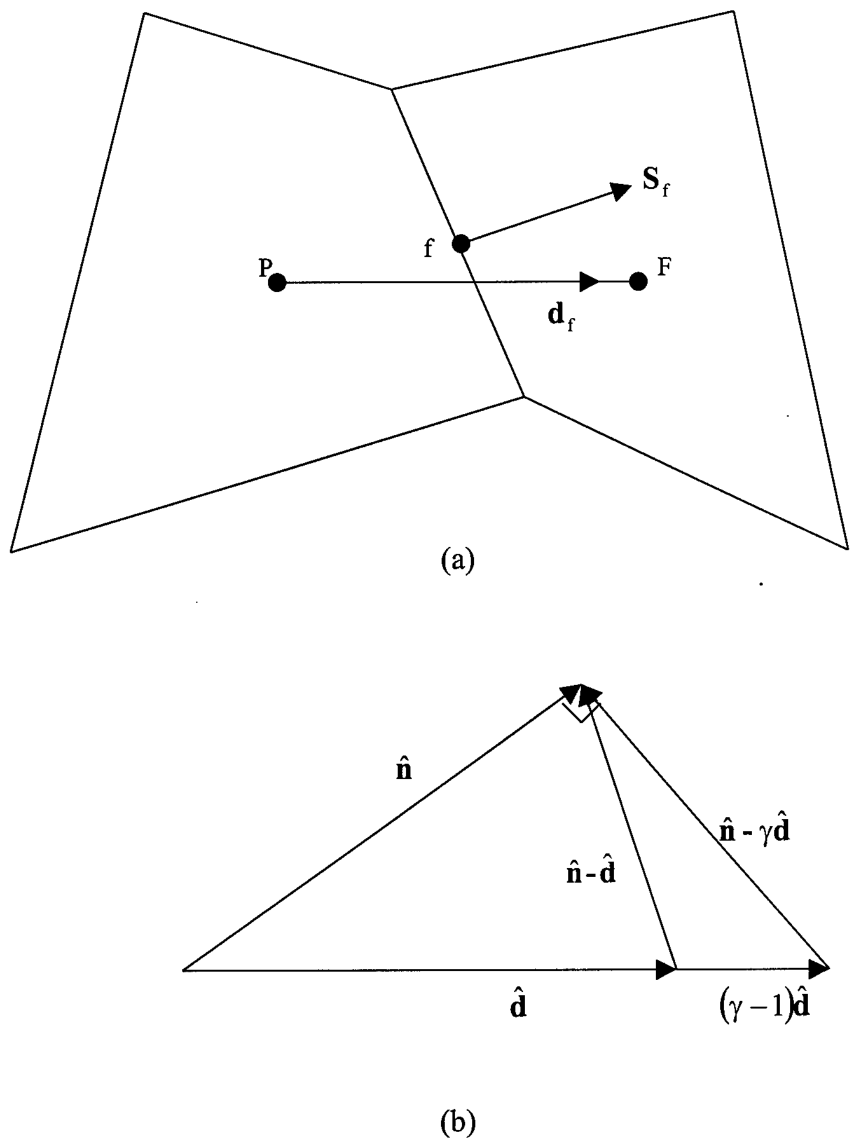

Fig. 2 Typical control volume faces and geometric nomenclature.